Deciding the On-line Chromatic Number of a Graph with Pre-Coloring is PSPACE-Complete

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On-line graph coloring

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- The algorithm assigns a color to v different from the colors it is already adjacent to. The color cannot be changed later.
- The goal is to use as few colors as possible.

An example

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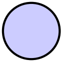

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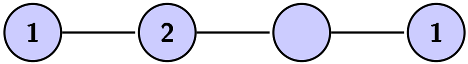

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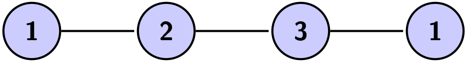


How to measure the quality of an algorithm?

Competitive Analysis [Sleator, Tarjan 85]

An algorithm A is c-competitive if there exists a b such that for any graph, G, the following holds:

$$A(G) \leq c \cdot \chi(G) + b$$

Here, $\chi(G)$ is the chromatic number of G. That is the minimum number of colors needed to color G (offline).

How to measure the quality of an algorithm?

Theorem [Gyarfas, Kiraly, Lehel 1990]

For any $k \in \mathbb{N}$, there exists a tree T_k such that any on-line algorithm can be forced to use k colors when on-line coloring T_k.

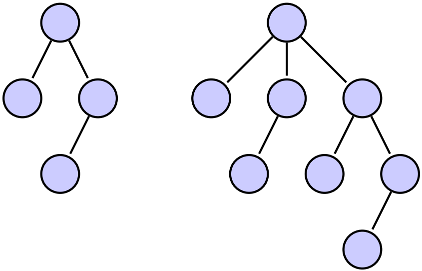


Figure: T_3 and T_4.

This means that no c-competitive algorithms exist for any constant c - even for the class of trees.

How to measure the quality of an algorithm?

On-line Competitive Analysis [Gyarfas, Kiraly, Lehel]

An algorithm A is on-line c-competitive if there exists an b such that for any graph, G, the following holds:

$$A(G) \leq c \cdot \chi^O(G) + b$$

Here, $\chi^O(G)$ is the on-line chromatic number of G. This is the smallest number such that there exists an algorithm that can color G using at most $\chi^O(G)$ colors for every ordering of the vertices.

On-line Chromatic Number

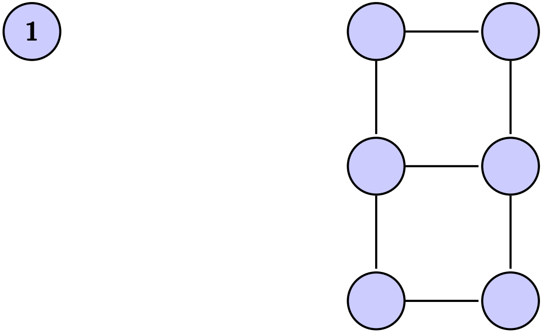
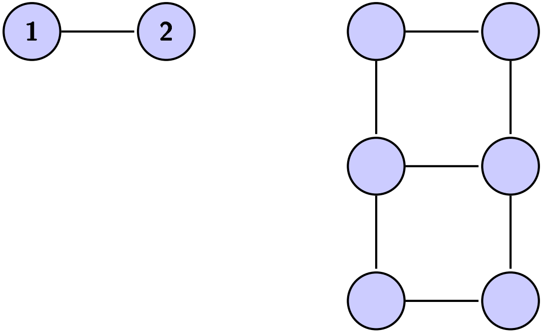


Figure: On-line coloring G. To the right, G is shown.

On-line Chromatic Number



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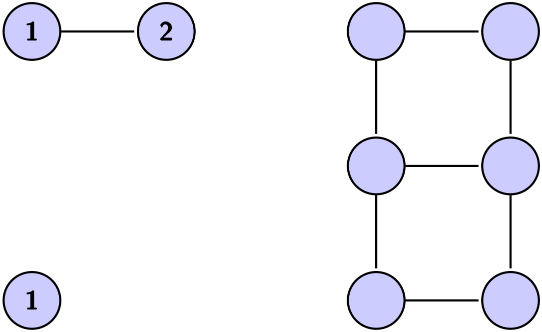

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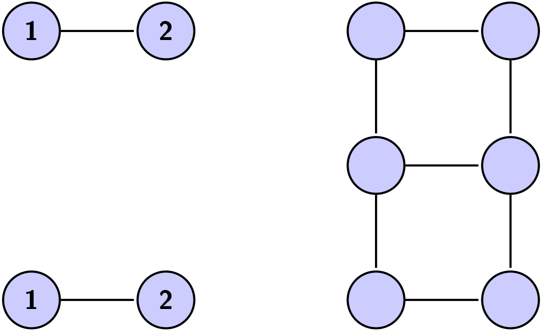

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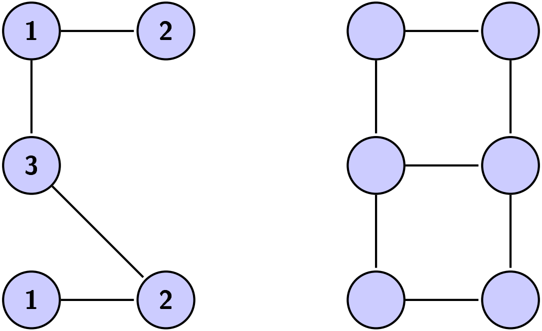

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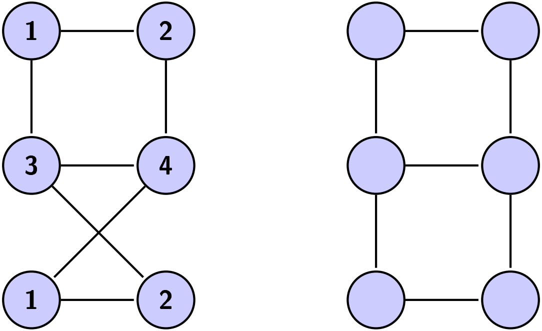

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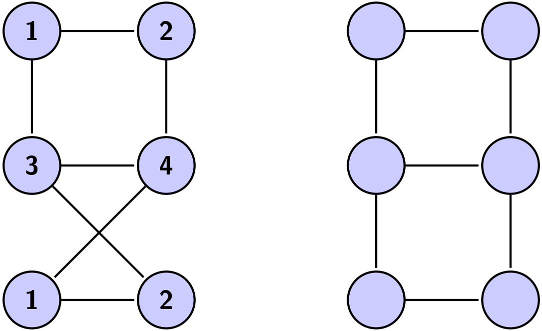

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It turns out that $\chi^O(G) = 3$ (the bottom right vertex should have been given the color 3 instead).

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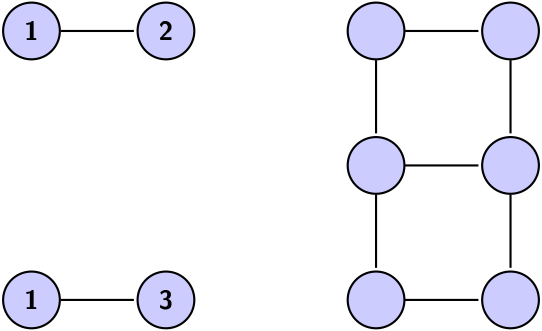

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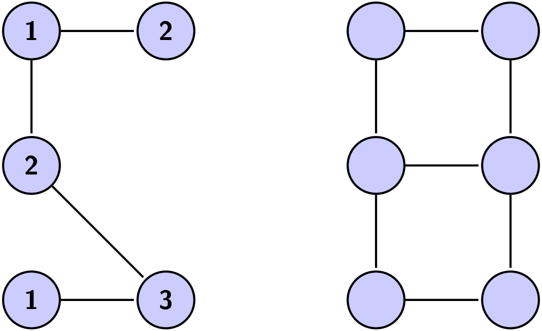

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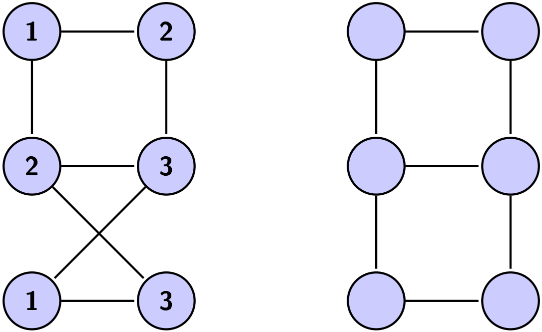

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We know that it is NP-complete given a $k \in \mathbb{N}$ and a graph G to decide if $\chi(G) \leq k$.

Main Problem

How hard is it given a $k \in \mathbb{N}$ and a graph G to decide if $\chi^O(G) \leq k$?

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- (G, k) is the game instance they agree on before the game.

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- At each time, the presented graph is isomorphic to an induced subgraph of G.
- If the painter is able to color the entire graph without using more than k colors, he wins. Otherwise, he loses.
- The painter has a winning strategy if and only if $\chi^O(G) \leq k$.

The On-line Chromatic Number

We can consider the following related problem:

On-line Chromatic Number with Pre-coloring

Given a state in the On-line Graph Coloring game, does the painter have a winning strategy from that state?

A state in the game means that a part of the graph has already been revealed and color.

This problem, which is also known as On-line Chromatic Number with Pre-coloring, is PSPACE-Complete.

Sketch of Proof

- Reduction from quantified satisfiability in 3-DNF.

$$F = \forall x_1 \exists x_2 \forall x_3 : (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge \overline{x}_3)$$

- Mapping F into graph G with precoloring and a k.

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- Mapping F into graph G with precoloring and a k.
- F true $\Leftrightarrow$ Painter can color G using at most k colors.
- F is false $\Leftrightarrow$ Drawer can force the painter to use more than k colors.

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- Mapping F into graph G with precoloring and a k.
- F true $\Leftrightarrow$ Painter can color G using at most k colors.
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- Two vertices for each variable.

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- One vertex for each term.

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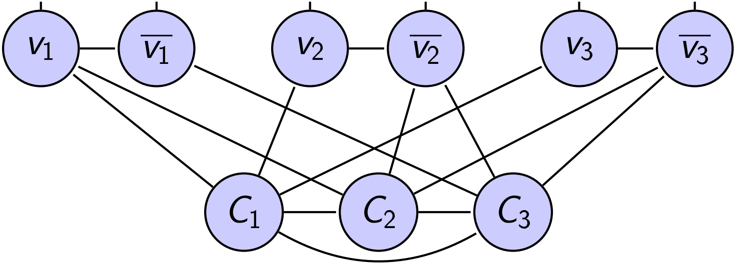

This is a part of the graph.

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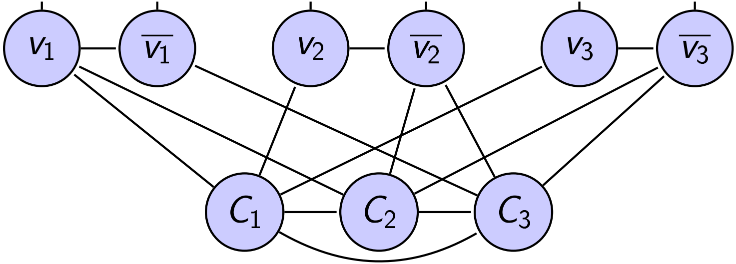

- Variables can only get color 'true' or 'false' (using pre-coloring)
- If C_i gets color 'false' (without conflict with variables), painter wins.
- Otherwise, he loses.

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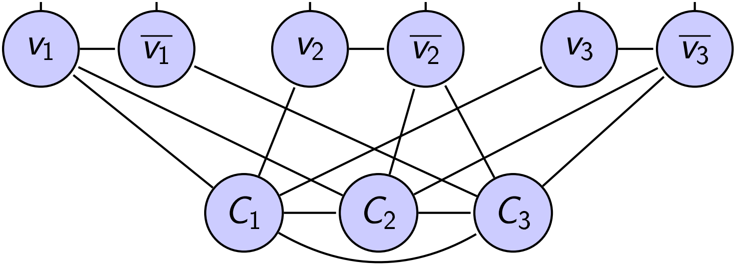

- When a variable or clause is requested, the painter knows the index (by pre-coloring).
- For the existentially quantified variables, the painter also know if it is v_i or $\overline{v_i}$.

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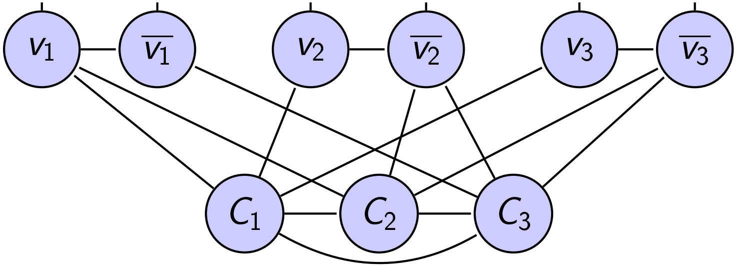

- If the formula is false, The Drawer can win.
- He requests the two vertices for each variable in order.
- For the universally quantified variables, he can decide the truth assignment.

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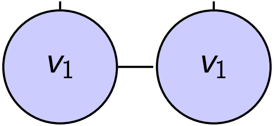

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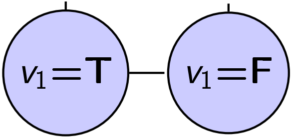

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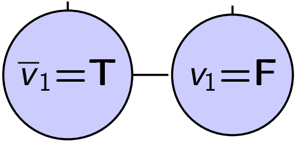

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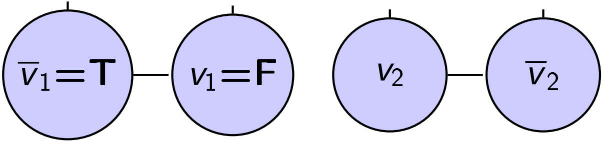

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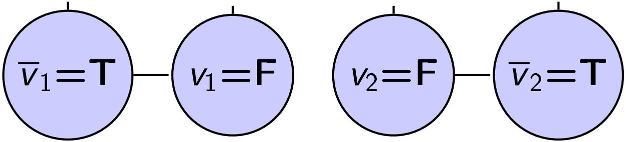

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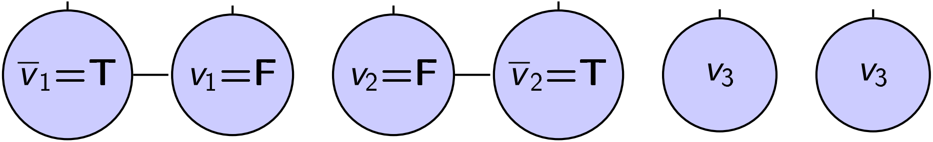

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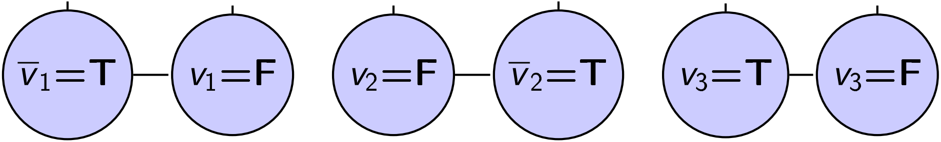

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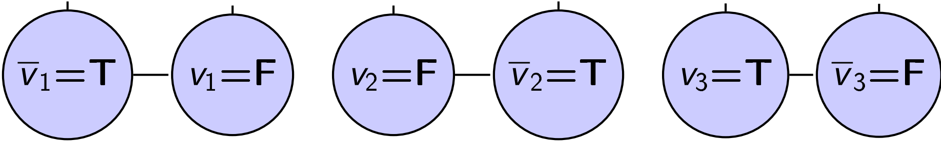

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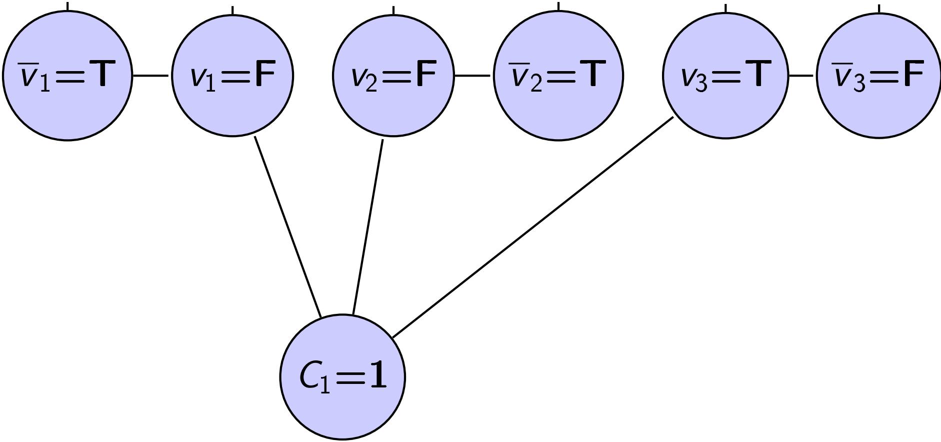

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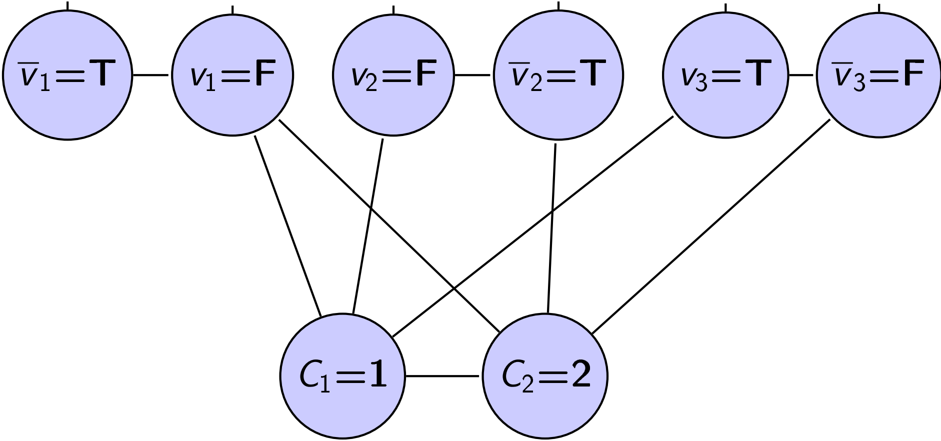

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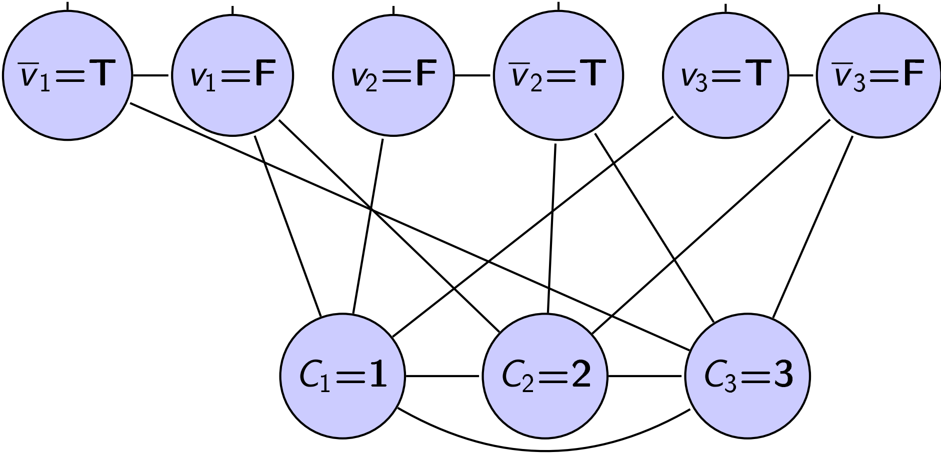

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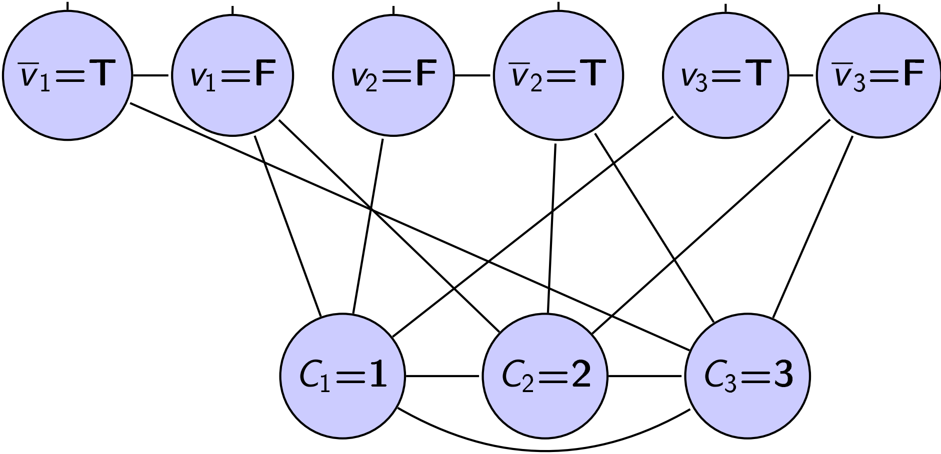

If one clause had been satisfied, it could have received the color false and the painter would have won.

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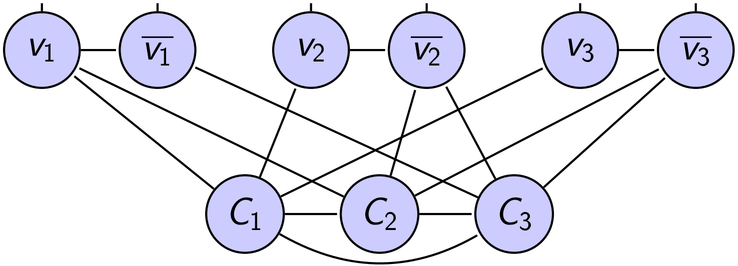

- An additional gadget ensures that the Drawer must request variables in order.
- When F is true, the Painter can color existentially quantified variables according to the 'correct' truth assignment from the satisfiability problem.
- One clause will have no neighbour with color false, so it can get that color.

Other results, Open Problems, and Conjectures

It is still unknown if the problem remains PSPACE-Complete without the pre-coloring.

Theorem

Online Chromatic Number without Pre-coloring is coNP-Hard.

Theorem

Online Chromatic Number without Pre-coloring is Σ_2^p-Hard in multigraphs.

Conjecture [Gyarfas, Lehel, Kiraly 98]

Given a graph, G, it is NP-hard to decide if $\chi^O(G) \leq 4$

Thank you for listening!