Competitive Analysis of the Online Inventory Problem

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Abstract

We consider a real-time version of the inventory problem with deterministic demand in which decisions as to when to replenish and how much to buy must be made in an online fashion without knowledge of future prices. We suggest online algorithms for each of four models for the problem and use competitive analysis to obtain algorithmic upper and lower bounds on the worst case performance of the algorithms compared to an optimal offline algorithm. These bounds are closely related to the tight $\sqrt{M/m}$-bound obtained for the simplest of the models, where M and m are the upper and lower bounds on the price fluctuation.

Key words: Inventory, Online algorithms, Competitive analysis

1. Introduction

We consider an inventory problem where the procurement price of a commodity is uncertain. In particular, we consider a set-up where the price fluctuates on a day to day basis, and decisions as to when and how much to buy have to be made in an online fashion, i.e., without any knowledge of future prices. Such problems often arise in dealing with raw materials. Our goal is to construct online algorithms to make decisions in such an environment and to prove bounds on their worst case performance. We compare our algorithms to an optimal offline algorithm using competitive analysis. In this way, we get a measure of the cost obtained by an algorithm compared to the optimal cost that could have been obtained if we had known all future prices in advance. This can be thought of as measuring the value of the information of future prices.

Deterministic inventory models and models with stochastic demand or stochastic lead time have been extensively studied in the literature; see [26, 32] for an overview. Models that takes into account the uncertainty of the various cost parameters are more rare. We give examples of such models below.

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Ben-Daya and Hariga [3], Chaudhuri and Ray [9], and Horowitz [20] take inflation into account. Akőz et al. [1] and Petrović et al. [27], among others, use fuzzy sets to model uncertainty of various cost parameters. Chaouch [8], Gurnani [18], and Moinzadeh [25] study models with two prices, where the lower price occurs at random points of time. Gurnani and Tang [19] consider a nested news vendor model with price uncertainty. Goel and Gutierrez [14] use Brownian motions to model two stochastic price streams based on different markets, such as spot markets and future contracts.

A number of papers study the problem of stochastic procurement price. Golabi [16] and Berling [4] study the problem where the price is drawn from a known distribution function. Wang [30] studies a set-up where prices are stochastically decreasing over time and demand is stochastic. Kingsman [23] studies the problem when demand is known and Kalymon [21] studies the problem with price dependency on previous prices where also demand is uncertain. Kouvelis and Li [24] analyze supply contracts in a model with stochastic prices.

In the next section, we introduce four inventory management models. Then we discuss the analytic methods we are using and present our results. In the following four sections, we treat the different models, after which we conclude.

2. Inventory Management Models

In a deterministic setting with constant price and constant withdrawal from the inventory, it is well known that the Economic Order Quantity (EOQ) is an optimal strategy [31]. The optimal order size is obtained by balancing the inventory storage costs on one side and the costs of placing orders on the other side. Doing that, it can be verified that the optimal order size is

$$Q = \sqrt{\frac{2 \ \times \ \text{order cost} \ \times \ \text{demand}}{\text{inventory cost per item per time unit}}}$$

We consider models which can be seen as variations of the EOQ model. We use t to represent time starting from initial time zero. It is assumed that the items are added to the inventory at times $1, 2, 3, \ldots$. We assume that the demand D is removed evenly during the time interval $[t, t+1)$. This corresponds to models studied in [28].

Without loss of generality, we scale everything according to D such that we may assume that $D = 1$. We also assume that any fractional amount of D can be identified. We refer to a quantity of one as an *item*, but emphasize that it is meaningful to purchase a fraction of an item.

We consider scenarios with varying prices, p_t, in the market, and assume fixed upper and lower bounds M and m, respectively, on p_t. The term Q_t denotes the quantity ordered at time t, and L_t denotes the inventory level at time t. If $Q_t \neq 0$, then we assume that L_t is the inventory level just before the purchase of Q_t items at time t. In algorithms, we often just specify the nonzero values of Q_t. We let S denote a fixed ordering cost. We use h to denote the inventory cost per item unit per time unit and let U denote the starting value of the inventory, i.e., $L_0 = U$. In some of our models, U will also be the maximum storage capacity. We assume that lead time is zero. This can be done without loss of generality because, since demand is deterministic, any delay between order and delivery can be incorporated by shifting the price sequence and adjusting the values of U and h.

We describe the models used in terms of restrictions on this general set-up. In each model, we either use bounded storage or holding costs as the limiting factor when prices are favorable. Our terminology is that the inventory at any given time refers to our total collection of items in a storage which may or may not have bounded capacity. We define the following four models.

The Bounded Storage Model. We assume a maximal storage capacity of U and assume that the inventory holding per item unit per time unit $h = 0$. Furthermore, we let the order cost $S = 0$. This model is studied in Section 4.

The Bounded Storage Order Cost Model. This model is as the Bounded Storage Model, but with $S > 0$. We study this model in Section 5.

The Unbounded Storage Model. In this model, we assume that the storage can hold an unlimited number of items at cost h per item per time unit. Items are removed evenly during the time interval $[t, t + 1)$. Thus, holding costs are larger in the beginning of that interval and smaller late in the interval, giving rise to a total holding cost of $h/2$ for the item we start removing at time t. We let the order cost $S = 0$. This model is studied in Section 6.

The Unbounded Storage Order Cost Model. This model is as the Unbounded Storage Model, but with $S > 0$. It is studied in Section 7.

3. Analytic Methods and Results

The aim is to design online algorithms with the best possible worst-case guarantees. We use the well-established technique of competitive analysis [17, 22, 29] to evaluate our design. An introduction to the technique can be found in [5]. Intuitively the idea is to compare the result obtained by an online algorithm to the result that could have been obtained if one had known all future prices in advance, where the latter scenario is represented by an optimal offline algorithm.

The basic idea of the technique is to find a minimum cost for the processing of a given sequence and use that as a reference point to compare algorithms up against. The problem we are studying is "online" in the sense that input (prices) are revealed one at a time and decisions must be made before the next input is seen. We define a hypothetical algorithm, usually referred to as OPT, which abbreviates "optimal". This algorithm knows the entire future and makes optimal decisions based on that. It still has to process each input one at a time. Clearly, no online algorithm can do better and OPT functions as the minimum cost reference point for evaluating all the online algorithms.

For an input sequence σ (a sequence of prices over a period of time), we let $ALG(\sigma)$ denote the cost of processing σ, and we are interested in the ratio $ALG(\sigma)/OPT(\sigma)$ for a given online algorithm ALG. Since we want to provide worst-case guarantees, we work on limiting the worst ratio taken over all input sequences. In order to avoid getting unrealistic results due to start-up costs for short sequences, we allow for an additive constant α, which is fixed and independent of σ, and define an algorithm ALG to be *c-competitive* if for all input sequences σ, $ALG(\sigma) \leq c \cdot OPT(\sigma) + \alpha$. The infimum over all such c is referred to as the *competitive ratio* of ALG and this number exactly characterizes its worst-case performance with regards to competitive analysis. Very often, it is too difficult to find the competitive ratio. Instead, one attempts to establish fairly close upper and lower bounds on the ratio.

Competitive analysis is a well-established method of analysis in computer science, and has also been used to study many problems within the areas of finance [10, 11, 12] and management science [6, 7, 13, 15].

The online inventory problem studied in this paper can be considered as an involved extension of the financial one-way trading problem [12] where one is concerned with the question as to when one should buy, i.e., in a sequence of prices arriving over time, one needs to pick a favorable one. Related work in [2] is concerned with picking a good option among several options that are available simultaneously, but have different properties that are revealed at a later time.

In the one-way trading problem, one assumes known minimum and maximum prices, m and M, respectively. Sometimes these can be given by bilateral agreements or law, but otherwise these must be estimated by experts. With this assumption, for the simplest problem of just making one purchase, it can be shown that the algorithm which accepts the first price better than or equal to $p^* = \sqrt{Mm}$ (called the reservation price) is $\sqrt{\frac{M}{m}}$-competitive. This is referred to as a reservation-price-policy [5].

The added complexity in the current paper stems partly from the introduction of a storage, the requirement that the inventory level in our storage is not allowed to drop below zero, and the necessity of restricting storage capacity upwards in some natural manner to avoid otherwise unrealistic results. Further complexity is brought on by the addition of an order cost parameter.

It is easy to verify that the strategy of filling the inventory to capacity (when a maximum is given) whenever the inventory is empty, results in a competitive ratio of $\frac{M}{m}$. In order to obtain better results, we start our exploration inspired by the reservation-price-policy, adapting our algorithms to the altered circumstances; the major change coming from the inventory restrictions that imply that the storage has to be replenished a potential unbounded number of times, depending on the length of the input sequence of daily prices.

We present upper and lower bounds on the competitive ratio of online algorithms for each of the four models. Our results are summarized in Figure 1.

For $S = 0$ and bounded storage, the result is tight. For unbounded storage, the result is harder to read because of the $\frac{h}{2}$-term. If one assumes that $\frac{h}{2}$ is insignificant compared to M and m, then the approximative bounds become $\frac{1}{2}\sqrt{\frac{M}{m}} \cdot (1 + \frac{1}{\sqrt{\frac{M}{m}}}) \lessapprox c(UA) \lessapprox \sqrt{\frac{M}{m}}$, i.e., there is less than a factor 2 between

	Bounded Storage	Unbounded Storage
$S = 0$	$c(BA) = \sqrt{\frac{M}{m}}$	$c(UA) \leq \sqrt{\frac{(M+\frac{h}{2})}{(m+\frac{h}{2})}} + \max\{\frac{2h}{h+2m}, \frac{h/2}{\sqrt{(m+h/2)(M+h/2)}+h/2}\}$ $c(UA) \geq \frac{1}{2}\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}} \cdot \frac{1+\frac{1}{\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}}}+\frac{h}{h+2M}}{1+\frac{h}{\sqrt{(m+\frac{h}{2})(M+\frac{h}{2})}}}$
$S > 0$	$c(BOA) \leq (2+\frac{1}{k})\sqrt{\frac{M}{m}}$ with $k = \frac{2-\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{\frac{M}{mU^2}+\frac{4\sqrt{\frac{M}{m}}}{U}+8}}{2+\frac{4\sqrt{\frac{M}{m}}}{U}}$ $c(BOA) \geq \frac{1}{2}\sqrt{\frac{M}{m}}$	$c(UOA) \leq \max\left\{\sqrt{\frac{M}{m}} + \frac{2\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 3, \sqrt{\frac{M}{m}}\left(\frac{M-m+h}{\sqrt{Sh}} + \frac{3}{\sqrt{2}}\right)\right\}$ $c(UOA) \geq \min\left\{\sqrt{\frac{M}{2m}}, \sqrt{\frac{S}{2h}}, \sqrt{\frac{M}{m}} \cdot \frac{1+\frac{h\sqrt{\frac{S}{h}}}{2M\sqrt{\frac{M}{m}}}+\frac{h}{2M}}{1+\frac{\sqrt{Sh}}{\sqrt{2Mm}}-\frac{\sqrt{Sh}}{2M}}\right\}$

Figure 1: Results for the algorithms *BA*, *UA*, *BOA*, and *UOA*.

the upper and lower bound. For $S > 0$ and bounded storage, the corresponding factor is a little more than 4, whereas in the last case, it is not really possible to talk about a factor independently of the concrete values, due to the complexity of the algorithm and result.

4. Bounded Storage Model

The following algorithm is inspired by the simple reservation-price-policy from the introduction and results in a significantly better competitive ratio than obtained through the naive strategy.

Bounded Storage Algorithm (*BA*) Let $p^* = \sqrt{mM}$. If at time t, we have $p_t \leq p^*$, order $Q_t = U - L_t$ items. Otherwise, if $p_t > p^*$ and $L_t = 0$, order one item.

Thus, when the price is better than p^*, we fill up to capacity, and otherwise we buy just enough to satisfy the removal from the inventory at the current time.

Figure 2 illustrates the behavior of *BA*. The upper diagram shows the item price over time, and the lower diagram shows the inventory level over time. At time t'', the price is below p^*, and we fill to capacity. After that the price is above p^*, and as long as $L_t > 0$, we do nothing. At time t', we have $L_t = 0$ and since the price is still above p^*, we buy just enough to satisfy the removal of one item at the current time. At time t^*, the price drops below p^*, so we fill up to capacity. As long as the price is above p^*, i.e., until time t^{**}, we fill to capacity. When the price rises again, we use the items in the inventory until the price drops below p^* at time $\hat{t}$.

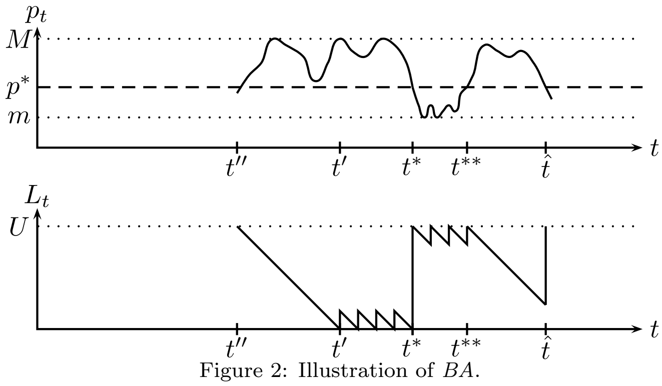

Figure 2: Illustration of BA.

The intuition behind BA is as follows. The price p^* is set to balance two situations that are bad for BA: BA and OPT pay M and p^*, respectively, and BA and OPT pay p^* and m, respectively. As long as the price is below p^*, the inventory is kept full, as the price will give us the balanced ratio. When the price it high, we only buy what is absolutely necessary. Since, if BA pays a high price, OPT will also have to pay at least p^* for those items, we will again obtain the balanced ratio. We formalize this argument in the following.

Theorem 1. BA has a competitive ratio of $\sqrt{\frac{M}{m}}$.

Proof We first show that BA is $\sqrt{\frac{M}{m}}$-competitive and establish the lower bound afterwards. To prove the competitiveness, we show the following.

1. If BA pays a price of more than p^*, then OPT will pay at least p^*.
2. If OPT pays a price below p^*, then BA pays no more than p^*.

If this holds for each item, then, per item, we have the worst case ratios $\frac{M}{\sqrt{mM}} = \sqrt{\frac{M}{m}}$ and $\frac{\sqrt{mM}}{m} = \sqrt{\frac{M}{m}}$, giving the desired competitive ratio.

We use the notation from Figure 2. First, assume that at some time, t', we have $L_{t'} = 0$ for BA, but the price is above p^*. In this situation, BA will be forced to buy at a high price. The only situation which can lead to $L_{t'} = 0$ at time t' is that the price has been above p^* for at least the past U time steps, say from time t'', since otherwise the inventory would have been filled to capacity during the period. Hence, OPT will also have been forced to buy at least one item at a price of at least $p^* + \varepsilon$ for some ε. For any time after t' where the price remains above p^*, the above argument can be extended to say that the two algorithms must buy the same number of items at a high price, i.e., above p^*. Therefore, the worst case for BA in this situation is that BA buys the items at a price of M, in which case OPT will buy exactly the same number of items at a price of at least $p^* + \varepsilon$ and the first statement is proven.

Since OPT must buy at least the same number of items at a price above p^* as BA, OPT can get at most the same number of items as BA at a price of at most p^*, which proves the second statement. For all these items, the worst price for BA is p^*, and the best possible price for OPT is m.

In adding up all the costs, it may be the case that BA has more items on hold than OPT after time n. This amounts to at most $U-1$ items, which gives an additive constant of at most $(U-1)p^*$.

Combining the two statements, we get $BA \leq \sqrt{\frac{M}{m}}OPT + (U-1)\sqrt{mM}$. Since the last term is a constant, this shows that BA is $\sqrt{\frac{M}{m}}$-competitive

To establish the lower bound, consider the sequence of prices starting with $p^*+\varepsilon$, followed by a price of M the next $U-1$ times; this construction repeated n times.

BA buys one item at each time at a cost of $n(p^*+\varepsilon+(U-1)M)$. OPT buys U items each time the price is $p^*+\varepsilon$ at a cost of $nU(p^*+\varepsilon)$. This gives

$$\frac{BA}{OPT} = \frac{p^*+\varepsilon+(U-1)M}{U(p^*+\varepsilon)} = \frac{M}{p^*+\varepsilon} + \frac{1-\frac{M}{p^*+\varepsilon}}{U}$$

Since $\varepsilon > 0$ can be chosen as close to zero as desirable and $\frac{1-\frac{M}{p^*}}{U}$ is a constant, BA cannot be c-competitive for any c smaller than $\frac{M}{p^*} = \sqrt{\frac{M}{m}}$. □

5. Bounded Storage Order Cost Model

Now, we add the order cost parameter to the previous model.

Let $k = \frac{2-\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{\frac{M}{mU^2}+\frac{4\sqrt{\frac{M}{m}}}{U}+8}}{2+\frac{4\sqrt{\frac{M}{m}}}{U}}$ and $b = \frac{U}{k\sqrt{\frac{M}{m}}}$.

Bounded Storage Order Cost Algorithm (*BOA*) Assume that $U \geq (\sqrt{2}-1)\sqrt{\frac{M}{m}}$. Let $p^* = \sqrt{Mm}$. If at time t, we have $p_t \leq p^*$ and $L_t \leq U-b$, order $Q_t = U - L_t$, i.e., fill up to capacity. If $p_t > p^*$ and $L_t < 1$, order up to level b.

Note that we must have $b \geq 1$, such that any given order will contain at least one item in order for the algorithm to be well-defined. This is equivalent to requiring that $\frac{\sqrt{\frac{M}{m}}}{U} \leq \frac{1}{k}$. We consider the following function which must then be less than or equal to zero.

$$
\begin{aligned}
\frac{\sqrt{\frac{M}{m}}}{U} - \frac{1}{k} &= \frac{\sqrt{\frac{M}{m}}}{U} - \frac{2+4\frac{\sqrt{\frac{M}{m}}}{U}}{2-\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{\frac{M}{mU^2}+4\frac{\sqrt{\frac{M}{m}}}{U}+8}} \\
&< \frac{\sqrt{\frac{M}{m}}}{U} - \frac{2+4\frac{\sqrt{\frac{M}{m}}}{U}}{2-\frac{\sqrt{\frac{M}{m}}}{U}+\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{8}} \\
&= \frac{\sqrt{\frac{M}{m}}}{U} - \frac{2+4\frac{\sqrt{\frac{M}{m}}}{U}}{2+\sqrt{8}} \\
&= \frac{(2+\sqrt{8})\frac{\sqrt{\frac{M}{m}}}{U}-2-4\frac{\sqrt{\frac{M}{m}}}{U}}{2+\sqrt{8}} \\
&= \frac{(\sqrt{8}-2)\frac{\sqrt{\frac{M}{m}}}{U}-2}{2+\sqrt{8}}
\end{aligned}
$$

This is less than or equal to zero if and only if $U \geq (\sqrt{2}-1)\sqrt{\frac{M}{m}}$. Note that this is not a very restrictive assumption in applications of the theory. If, for example, the ratio between M and m is as large as 100, the requirement is that the inventory must have space for items equal to the demand of 4.2 time units. For more realistic ratios between M and m, the requirement is even smaller.

As indicated in Figure 3, we define a *phase* to be from the time the price drops below p^* to the next time the price drops below p^*, i.e., a phase will consist of a period with low prices, followed by a period with high prices. We consider the cost incurred by each of the algorithms, BOA and OPT, based on these phases.

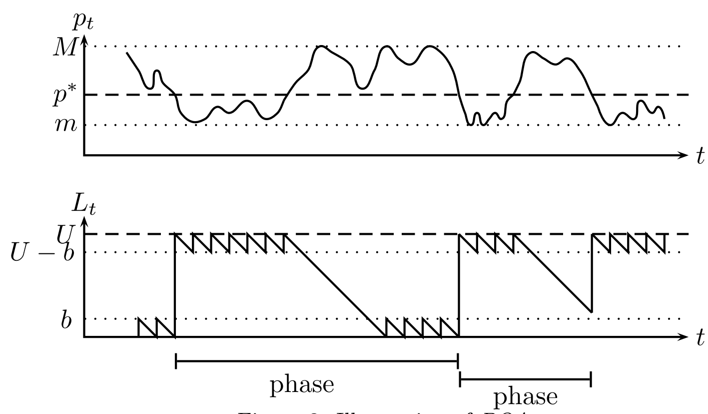

Figure 3: Illustration of BOA.

The intuition behind BOA is very similar to that of BA. The price p^* is again determined by balancing the two extreme situations. However, the difference from the situation with BA is that BOA must pay an additional cost each time it places an order. Thus, it is not desirable to order single items, even when the price is high. For this reason, b items are bought in the cases where BA only bought one. In order to determine the magnitude of b, we balance the sum of the item costs for BOA and OPT, respectively, and the total ordering costs for each. For the latter, the order costs cancel out when considering the ratio, and the determining part becomes the number of orders placed; not the cost of each order. During the balancing, the value of k is determined.

We define $\Delta = k\sqrt{\frac{M}{m}}$ and hence, we have $b = \frac{U}{\Delta}$. Intuitively, Δ is the number of replenishments it takes to buy U items if b items are bought each time. We assume that $\Delta \geq 1$.

Theorem 2. When $U \geq k\sqrt{\frac{M}{m}} \geq 1$, BOA is $\left(2 + \frac{1}{k}\right)\sqrt{\frac{M}{m}}$-competitive.

Proof For the analysis, define x and y to be time intervals. When the inventory goes from full to empty, U items are removed, which corresponds to a time period of U. This is illustrated in Figure 4, where the inventory level of a single phase is shown.

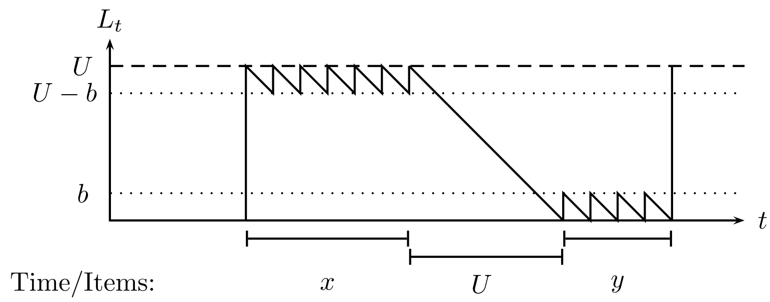

Figure 4: Illustration of one phase for BOA.

First, we establish a lower bound on the cost of OPT. The total number of items removed from the inventory in the phase is $x+y+U$. Since at best, OPT can buy U items each time, OPT will have at least the following order cost:

$$OPT_o \geq \frac{S(x+y+U)}{U}$$

For OPT, a lower bound on the total cost of the items that are removed in the phase is given by

$$OPT_i \geq mx + mU + m\tau + p^*(y - \tau)$$

where τ is the number of items removed from the last time BOA replenished at low price until the price goes up. By the definition of BOA, we must have $\tau \leq b$. During the period of length τ, OPT can buy items at a price of m, while BOA cannot buy for a time period of length b, implying that it will have to buy at the price M some U time steps later, while OPT can use items acquired at the price m. Thus, as a worst-case assumption, we can assume that $\tau = b$. Hence,

$$OPT_i \geq mx + mU + mb + p^*(y - b)$$

Next, we derive an upper bound on the cost of BOA. For BOA, the maximal price payed for items that are removed from the inventory in the phase is

$$BOA_i \leq p^*(x+U) + My$$

In the low price part of the phase (excluding the first time the inventory is replenished), BOA replenishes $\left\lfloor \frac{x}{\lceil b \rceil} \right\rfloor$ times and buys $\lceil b \rceil$ items each time. In the high price part of the phase, BOA replenishes at most $\frac{y}{b-1}$ times, since there are y items and BOA buys at least $b-1$ items each time (it buys $b - L_t$ items whenever $L_t < 1$). Finally, the last time, where the inventory is filled up for the use in the next phase, must be counted. In total, the order cost of BOA is bounded by

$$BOA_o \leq S\left(\frac{x}{b} + \frac{y}{b-1} + 1\right)$$

In order to simplify the below calculations, we define $\Psi = 1 - 1/b$. We determine the best value for k which will turn out to produce the value for Δ, and therefore also b, which is used in the definition of BOA.

We can now write the costs as follows:

Costs	Order	Items
BOA	$S\left(\frac{x}{b} + \frac{y}{b-1} + 1\right)$	$p^*(x+U) + My$
OPT	$S\frac{x+y+U}{U}$	$m\left(x + U + \frac{U}{\Delta}\right) + p^*\left(y - \frac{U}{\Delta}\right)$

We consider the competitive ratio of BOA separately with regards to the order and item costs. The maximum of these is a valid upper bound on the combined competitive ratio.

First we consider the item costs of BOA compared with OPT. In the first inequality below, we use that with regards to item cost, in the worst case, $y \geq \tau = b = \frac{U}{\Delta}$. We set $y' = y - \frac{U}{\Delta} \geq 0$ and use the values of p^* and Δ. We use several times that for all positive a, b, c, d, $\frac{a+b}{c+d} \leq \frac{a}{c} + \frac{b}{d}$.

$$\begin{aligned}
\frac{BOA_i}{OPT_i} &= \frac{p^*(x+U)+My}{m(x+U+\frac{U}{\Delta})+p^*(y-\frac{U}{\Delta})} = \frac{p^*(x+U)+My}{mx+\left(1+\frac{1}{\Delta}\right)mU+p^*\left(y-\frac{U}{\Delta}\right)} = \frac{p^*(x+U)+My'+M\frac{U}{\Delta}}{m(x+U)+p^*y'+m\frac{U}{\Delta}} \\
&= \frac{p^*x+My'+U\left(p^*+\frac{M}{\Delta}\right)}{mx+p^*y'+U\left(m+\frac{m}{\Delta}\right)} \leq \frac{p^*x+My'}{mx+p^*y'} + \frac{U\left(p^*+\frac{M}{\Delta}\right)}{U\left(m+\frac{m}{\Delta}\right)} = \sqrt{\frac{M}{m}} + \frac{p^*+\frac{M}{\Delta}}{m+\frac{m}{\Delta}} \\
&\leq \sqrt{\frac{M}{m}} + \frac{p^*+\frac{M}{k\sqrt{\frac{M}{m}}}}{m+\frac{m}{\Delta}} = \sqrt{\frac{M}{m}} + \frac{p^*+\frac{1}{k}p^*}{m+\frac{m}{\Delta}} = \sqrt{\frac{M}{m}} + \sqrt{\frac{M}{m}}\left(\frac{1+\frac{1}{k}}{1+\frac{1}{\Delta}}\right) \\
&\leq \sqrt{\frac{M}{m}}\left(2+\frac{1}{k}\right)
\end{aligned}$$

Next, we consider the ordering costs of BOA compared with OPT. We use that $(U-\Delta)/U = 1 - \Delta/U = 1 - 1/b$ as well as $\Psi < 1$ and $\Delta \geq 1$.

$$\frac{BOA_o}{OPT_o} = \frac{S(\frac{x}{b}+\frac{y}{b-1}+1)}{S\frac{x+y+U}{U}} = \frac{\frac{x}{U/\Delta}+\frac{y}{U/\Delta-U/U}+U/U}{\frac{x+y+U}{U}} = \frac{\frac{x}{1/\Delta}+\frac{y}{1/\Delta-1/U}+U}{x+y+U}$$

$$= \Delta\frac{x+\frac{y}{(1/\Delta-1/U)\Delta}+\frac{U}{\Delta}}{x+y+U} = \Delta\frac{x+\frac{y}{(U-\Delta)/U}+\frac{U}{\Delta}}{x+y+U} = \Delta\frac{x+\frac{y}{\Psi}+\frac{U}{\Delta}}{x+y+U} = \frac{\Delta}{\Psi}\frac{x\Psi+y+\frac{U\Psi}{\Delta}}{x+y+U}$$

$$\leq \frac{\Delta}{\Psi}\frac{x+y+U}{x+y+U} = \frac{\Delta}{\Psi} = \frac{k\sqrt{\frac{M}{m}}}{(1-1/b)}$$

In balancing the bounds from the two cases, we solve $2+\frac{1}{k} = \frac{k}{1-1/b}$. Substituting $\frac{U}{k\sqrt{\frac{M}{m}}}$ in the place of b gives $2+\frac{1}{k} = \frac{k}{1-\frac{k\sqrt{\frac{M}{m}}}{U}}$ or equivalently $(1+\frac{2\sqrt{\frac{M}{m}}}{U})k^2+(\frac{\sqrt{\frac{M}{m}}}{U}-2)k-1=0$.

The only positive solution to this equation is

$$k = \frac{2-\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{(\frac{\sqrt{\frac{M}{m}}}{U}-2)^2+4(1+\frac{2\sqrt{\frac{M}{m}}}{U})}}{2(1+\frac{2\sqrt{\frac{M}{m}}}{U})} = \frac{2-\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{\frac{M}{mU^2}+\frac{4\sqrt{\frac{M}{m}}}{U}+8}}{2+\frac{4\sqrt{\frac{M}{m}}}{U}}$$

Hence, choosing this value for k gives the best result.

In total, the ratio between the costs for BOA and OPT is bounded by

$$\left(2+\frac{2+\frac{4\sqrt{\frac{M}{m}}}{U}}{2-\frac{\sqrt{\frac{M}{m}}}{U}+\sqrt{\frac{M}{mU^2}+\frac{4\sqrt{\frac{M}{m}}}{U}+8}}\right)\sqrt{\frac{M}{m}}$$

Both OPT and BOA start with a full inventory, but it is possible that OPT will end with fewer items in the inventory than BOA. This will result in an additive constant of at most $S+\max\{Up^*, bM\}$. □

We now give a lower bound on the competitive ratio of BOA.

Theorem 3. The competitive ratio of BOA is at least $\frac{1}{2}\sqrt{\frac{M}{m}}$.

Proof The idea of the proof is illustrated in Figure 5. We consider a phase of length $3U$. The phase of OPT is shifted to the right by $b-\varepsilon$ compared to the phase of BOA. This shift will result in an additive constant and can thus be ignored in the following. The phase is repeated as many times as desired. During the phase, the price sequence is as follows:

price	time interval
p^*	$[\ t_1\ ,\ t_1+b-\varepsilon\ )$
m	$t_1+b-\varepsilon$
p^*	$(\ t_1+b-\varepsilon\ ,\ t_2+b-\varepsilon\ )$
m	$t_2+b-\varepsilon$
M	$(\ t_2+b-\varepsilon\ ,\ t_3+b-\varepsilon\ )$
$p^*+\mu$	$t_3+b-\varepsilon$
M	$(\ t_3+b-\varepsilon\ ,\ t_4=t_1\ )$

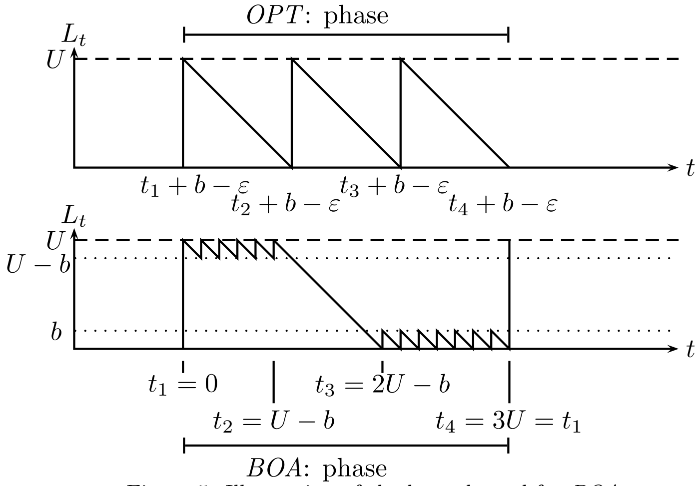

Figure 5: Illustration of the lower bound for BOA.

We consider the ratio of BOA to OPT separately with regards to item cost and ordering cost. The smaller of the two will be a lower bound on the competitive ratio of BOA. We consider the limit as ε and μ go towards zero.

When going through the construction below, it is easy to see that if b does not divide U, this can only lead to an extra order, and the time interval where BOA buys at the price M will be longer, so costs will be higher. Thus, in establishing a lower bound, we may assume that b divides U.

In each phase, OPT orders 3 times and BOA orders $2\frac{U}{b} + 1$ times, giving the ratio $\frac{BOA_o}{OPT_o} = \frac{2\frac{U}{b}+1}{3} = \frac{2}{3}\Delta + \frac{1}{3} = \frac{2}{3}k\sqrt{\frac{M}{m}} + \frac{1}{3}$.

In each phase, the item cost for OPT is $2Um + Up^*$ and the item cost for BOA is $p^*(2U - b) + M(U + b)$.

The item cost ratio is therefore

$$\frac{BOA_i}{OPT_i} = \frac{p^*(2U - b) + M(U + b)}{2Um + Up^*}$$

We now show that this ratio is at least $\frac{1}{2}\sqrt{\frac{M}{m}}$. In the first step, we divide by Up^*.

$$\begin{array}{rcl} p^*(2U - b) + M(U + b) & \geq & \frac{1}{2}\sqrt{\frac{M}{m}}(2Um + Up^*) \\ \Updownarrow & & \\ 2 - \frac{b}{U} + \sqrt{\frac{M}{m}} + \frac{b}{U}\sqrt{\frac{M}{m}} & \geq & 1 + \frac{1}{2}\sqrt{\frac{M}{m}} \\ \Updownarrow & & \\ 1 + (\sqrt{\frac{M}{m}} - 1)\frac{b}{U} + \frac{1}{2}\sqrt{\frac{M}{m}} & \geq & 0 \end{array}$$

which clearly holds since $M > m$.

Since the other bound of $\frac{2}{3}k\sqrt{\frac{M}{m}} + \frac{1}{3}$ is always greater than $\frac{1}{2}\sqrt{\frac{M}{m}}$, for $k \geq 1$, the result follows. □

6. Unbounded Storage Model

We replace the maximum inventory level by an inventory holding cost per item per time unit, h.

Unbounded Storage Algorithm (*UA*) Let $p^* = \sqrt{(m+\frac{h}{2})(M+\frac{h}{2})} + \frac{h}{2}$ and for any time t, let $Q'_t = 2\frac{p^*-p_t}{h} - 2L_t + 1$ and order

$$Q_t = \begin{cases} Q'_t & \text{if } Q'_t > 0 \text{ and } L_t + Q'_t \geq 1 \\ 1 - L_t & \text{if } L_t + \max\{0, Q'_t\} < 1 \\ 0 & \text{otherwise} \end{cases}$$

The intuition behind *UA* is as follows. *UA* replenish in two situations: When the price and the inventory level is relatively low, such that the average cost of items bought are below a pre-specified threshold value, and when it is necessary in order to satisfy the demand in the following period. In the former case, intuitively we would like to buy as many items as possible because they are cheap. However, inventory costs grow (quadratically) in the number of items purchased. We compute the total cost as a function of how many items we buy and determine the number of items to buy (Q'_t) to be the largest possible number we can buy without exceeding the desired average cost of $p^*+\frac{h}{2}$. Finally, the value of p^* is obtained by considering the relative costs of our algorithm compared to OPT separately for each case, and then balancing these to minimize the maximum. In that way, we obtain the best possibly worst-case guarantee obtainable using this approach.

In the unbounded storage models, holding cost could lead to the purchase of fractional parts of items. In the competitive analyses dealing with unbounded storage, it is helpful to know that this option can never be to the advantage of OPT, which we will prove below.

In the following, we let $\overline{p}$ be the total cost of an item defined as the price of the item plus the holding cost for the period from the item is bought until it is removed from the inventory. By definition, if an item bought at time $t-i$ is removed at time t, its item price is p_{t-i}. Its holding cost is $ih + f\frac{h}{2}$, where f, $0 < f \leq 1$, depends on when during the time $[t, t+1)$ the possibly fractional item is removed.

The following lemma is used in the proof of Theorem 4 below.

Lemma 1. There exists an optimal algorithm that does not buy items in fractional parts.

Proof Consider the item removed from the inventory by OPT at time t and assume that this item was bought in fractional parts $r_j, j = 1 \ldots k$, with $\sum_{j=1}^{k} r_j = 1$. Assume that the fraction r_j was bought at time $t - i_j$. The cost of the item is

$$\overline{p} = \left(\sum_{j=1}^{k} r_j(p_{t-i_j} + i_j h)\right) + h\left(\frac{1}{2}r_1^2 + \sum_{j=2}^{k}\left(\frac{1}{2}(\sum_{l=1}^{j} r_l)^2 - \frac{1}{2}(\sum_{l=1}^{j-1} r_l)^2\right)\right)$$

where the first part is the item price plus the holding cost from the time the item is bought to the beginning of time t, and the last part is the holding cost in the period from t to $t+1$. The latter contains a telescoping sum which can be reduced as follows:

$$\begin{aligned}\overline{p} &= \left(\sum_{j=1}^{k} r_j(p_{t-i_j} + i_j h)\right) + h\left(\frac{1}{2}r_1^2 + \frac{1}{2}\left(\sum_{l=1}^{k} r_l\right)^2 - \frac{1}{2}\left(\sum_{l=1}^{1} r_l\right)^2\right) \\ &= \sum_{j=1}^{k} r_j(p_{t-i_j} + i_j h) + \frac{1}{2}h\end{aligned}$$

The above calculation is illustrated in Figure 6 for $k = 4$.

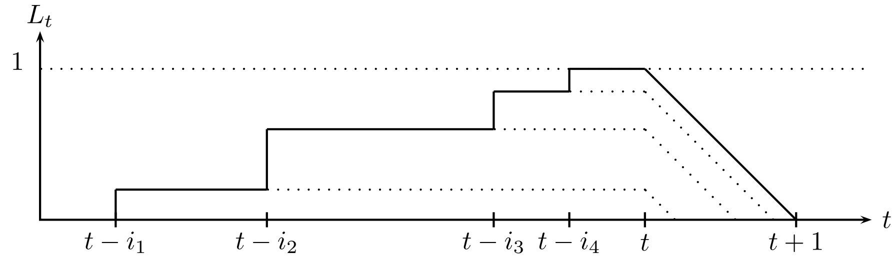

Figure 6: Illustration of fractional holding costs.

Define j_{min} by $j_{min} = \min_j\{p_{t-i_j} + i_j h\}$. So for all j, we have $p_{t-i_{j_{min}}} + i_{j_{min}} h \leq p_{t-i_j} + i_j h$. Hence, the cost of an item bought at time $t - i_{j_{min}}$,

$$\begin{aligned}\overline{p_{t-i_{j_{min}}}} &= p_{t-i_{j_{min}}} + i_{j_{min}} h + \frac{1}{2}h = \sum_{j=1}^{k} r_j(p_{t-i_{j_{min}}} + i_{j_{min}} h) + \frac{1}{2}h \\ &\leq \sum_{j=1}^{k} r_j(p_{t-i_j} + i_j h) + \frac{1}{2}h = \overline{p}\end{aligned}$$

Thus, it will never be cheaper for OPT to buy an item in fractional parts. □

We now state an upper bound on the competitive ratio of UA. When analyzing costs on fractional parts, we use the fact that OPT can be assumed not to buy in fractional parts. We can then allow ourselves to spread the holding cost of one item used by OPT in a time period $[t, t+1)$ out evenly. More precisely, instead of having a larger holding cost for a fractional item used late in $[t, t+1)$ than for one used early in $[t, t+1)$, we assign an average holding cost of $\frac{h}{2}$ to all fractional items used by OPT.

Theorem 4. *UA is* $\left(\sqrt{\frac{(M+h/2)}{(m+h/2)}} + \max\left\{\frac{2h}{h+2m}, \frac{h/2}{\sqrt{(m+h/2)(M+h/2)}+h/2}\right\}\right)$*-competitive.*

Proof We show that for each item, either the average cost UA pays is the same as for OPT or the following holds:

1. If UA orders Q'_t, then the average cost of these items is at most $p^* + \frac{h}{2}$.
2. If UA orders Q'_t, then OPT's average cost is at least $m + \frac{h}{2}$ for these items.
3. If UA orders $1 - L_t$, then the average cost of these items is at most $M + h$.

4. If UA orders $1 - L_t$, then OPT's average cost is at least p^* for these items.

The four statements above imply that $\frac{UA}{OPT} \leq \max\left\{\frac{p^*+h/2}{m+h/2}, \frac{M+h}{p^*}\right\}$. Since

$$\frac{p^* + h/2}{m + h/2} = \frac{\sqrt{(m + h/2)(M + h/2)}}{m + h/2} + \frac{h}{m + h/2} = \sqrt{\frac{M + h/2}{m + h/2}} + \frac{2h}{h + 2m}$$

and $\frac{M+h}{p^*} = \frac{M+h/2+h/2}{\sqrt{(m+h/2)(M+h/2)}+h/2} < \sqrt{\frac{M+h/2}{m+h/2}} + \frac{h/2}{\sqrt{(m+h/2)(M+h/2)}+h/2}$,the four statements imply an upper bound on the competitive ratio of

$$\sqrt{\frac{M + h/2}{m + h/2}} + \max\left\{\frac{2h}{h + 2m}, \frac{h/2}{\sqrt{(m + h/2)(M + h/2)} + h/2}\right\}$$

We now establish the four statements.

Statement 1. At time t, the inventory level is L_t, and UA buys Q_t items. The total cost of the Q_t items is

$$\frac{1}{2}h(Q_t + L_t)^2 - \frac{1}{2}hL_t^2 + Q_t p_t = h(\frac{1}{2}Q_t^2 + Q_t L_t) + Q_t p_t$$

The average cost of the items is

$$\begin{aligned} \frac{h(\frac{1}{2}Q_t^2+Q_tL_t)+Q_tp_t}{Q_t} &= \frac{h}{2}Q_t + hL_t + p_t \\ &= \frac{h}{2}(2\frac{p^*-p_t}{h} - 2L_t + 1) + hL_t + p_t = p^* + \frac{h}{2} \end{aligned}$$

Statement 2. Obvious, since m is the minimum price payed for any item and OPT's holding cost is spread out evenly.

Statement 3. Obvious, since M is the maximal price of an item and at most one item is bought each time.

Statement 4. In this case, the size of UA's inventory is smaller than one, the price is relatively high, and UA is forced to buy a fraction of an item which is more than what is given by Q'_t, i.e., $L_t + Q'_t < 1$.

We consider the item that OPT removes from the inventory at this time. By Lemma 1, we may assume that for some i, OPT bought this item at time $t - i$. If for OPT, $\overline{p} \geq p^*$, we are done, so assume to the contrary that $\overline{p} < p^*$.

The intuitive idea is that then UA would have bought enough items at time $t - i$ to be able to remove one of these at time t, contradicting the fact that UA's inventory is below one. We now formalize this argument.

OPT's total cost for this item is $\overline{p} = p_{t-i} + ih + \frac{1}{2}h = p_{t-i} + \frac{2i+1}{2}h$. By the assumption that $\overline{p} < p^*$, we get that $p_{t-i} < p^* - \frac{2i+1}{2}h$. This means that $Q'_{t-i} > 2\frac{p^*-(p^*-\frac{2i+1}{2}h)}{h} - 2L_{t-i} + 1 = 2(i+1) - 2L_{t-i}$.

If $L_t < 1$, then $L_{t-i} < i + 1$. Thus, $Q'_{t-i} > 0$, and at least Q'_{t-i} items were bought at time $t - i$. This would lead to an inventory at time t of at least $L_{t-i} + (2(i+1) - 2L_{t-i}) - i = i + 2 - L_{t-i} \geq 1$, contradicting that $L_t < 1$. □

In the following lower bound, for comparison with the upper bound, note that in the last factor, both the enumerator and the denominator are between one and three, and, for M going towards infinity, the last factor approaches one.

Theorem 5. The competitive ratio of UA is at least $\frac{1}{2}\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}}\cdot\frac{1+\frac{1}{\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}}}+\frac{h}{h+2M}}{1+\frac{h}{\sqrt{(m+\frac{h}{2})(M+\frac{h}{2})}}}$.

Proof Let the price sequence be $p_t = p^* - \frac{h}{2}$ for all odd values of t and $p_t = M$ for all even values of t.

At time 1, we have $Q'_t = 2\frac{p^*-(p^*-\frac{h}{2})}{h} - 2L_t = 1$, so UA buys one item. At time 2, Q'_t is clearly negative, but since the inventory is empty, UA is forced to buy one item. This continues n times, giving a total cost for UA of $n(p^*-\frac{h}{2}+\frac{h}{2}+M+\frac{h}{2}) = n(p^*+M+\frac{h}{2})$.

For sufficiently large values of M, OPT will buy two items for all odd values of t. The total cost for OPT is therefore $n(2(p^*-\frac{h}{2})+\frac{1}{2}h+\frac{3}{2}h) = n(2p^*+h)$.

This gives the following ratio between UA and OPT:

$$\begin{aligned}\frac{UA}{OPT} &= \frac{n(p^*+M+\frac{h}{2})}{n(2p^*+h)} = \frac{1}{2}\frac{M+\frac{h}{2}+(p^*-\frac{h}{2})+\frac{h}{2}}{p^*-\frac{h}{2}+h} = \frac{1}{2}\frac{\frac{M+\frac{h}{2}}{p^*-\frac{h}{2}}+1+\frac{h}{2(p^*-\frac{h}{2})}}{1+\frac{h}{p^*-\frac{h}{2}}} \\ &= \frac{1}{2}\frac{\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}}+1+\frac{h}{2\sqrt{(m+\frac{h}{2})(M+\frac{h}{2})}}}{1+\frac{h}{\sqrt{(m+\frac{h}{2})(M+\frac{h}{2})}}} = \frac{1}{2}\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}}\cdot\frac{1+\frac{1}{\sqrt{\frac{M+\frac{h}{2}}{m+\frac{h}{2}}}}+\frac{h}{h+2M}}{1+\frac{h}{\sqrt{(m+\frac{h}{2})(M+\frac{h}{2})}}}\end{aligned}$$

□

7. Unbounded Storage Order Cost Model

We add an order cost parameter to the model of the previous section.

Unbounded Storage Order Cost Algorithm (*UOA*) Let $b = \frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 1$, $p^* = \sqrt{Mm} + h\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 1\right)$, and for any time t, let $Q'_t = 2\frac{p^*-p_t}{h} - 2L_t + 1$ and order

$$Q_t = \begin{cases} Q'_t & \text{if } Q'_t \geq b \\ b - L_t & \text{if } Q'_t < b \text{ and } L_t < 1 \\ 0 & \text{otherwise} \end{cases}$$

The algorithm is illustrated in Figure 7. In the algorithm there are two situations: In Situation 1, we replenish because it is attractive to do so due to the current price and inventory level. In Situation 2, we replenish because we are forced to do so due to low inventory level.

The intuition behind UOA is very similar to that of UA, but also with elements from BOA. When we replenish because it is attractive to do so, we

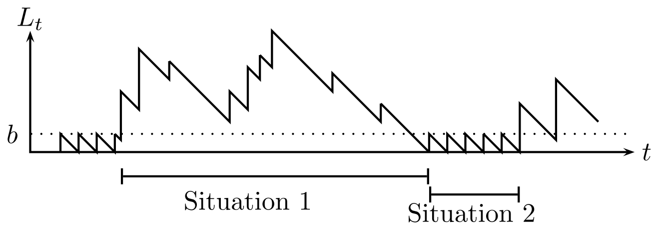

Figure 7: Illustration of *UOA*.

order exactly enough to obtain an average cost of the items equal to a prespecified threshold value. This order quantity is given by Q'_t. When we are forced to replenish in an unattractive situation, we must consider the ordering cost as we did for *BOA*, and set the order quantity higher than we would have done if there were no order cost. Thus, never ordering less than a certain bound, b. In each of the two situations, we consider the ratio of the cost of items bought by *UOA* to the cost of items bought by *OPT*. These two ratios are finally balanced against the ordering costs obtained by *UOA* and *OPT*, respectively, in order to obtain the values of p^*, b, and Q'_t, which are the parameters of the algorithm.

In the following, we analyze the two situations separately considering also item cost and order cost separately. The final result is thus the greater of the two.

Theorem 6. *UOA* is $\max\left\{\sqrt{\frac{M}{m}} + \frac{2\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 3, \sqrt{\frac{M}{m}}\left(\frac{M-m+h}{\sqrt{Sh}} + \frac{3}{\sqrt{2}}\right)\right\}$-competitive.

Proof First, we analyze Situation 1. Each time *UOA* replenishes, the average cost, including both item price and holding cost, of the items is exactly $p^* + h/2$. The analysis of this is identical to the analysis of Statement 1 in the proof of Theorem 4, where it is also argued that we can set *OPT*'s cost to at least $m + h/2$. Hence, in Situation 1 we have the following ratio with respect to item cost:

$$\frac{p^*+\frac{h}{2}}{m+\frac{h}{2}} \quad = \quad \frac{\sqrt{Mm}+h\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)+\frac{h}{2}}{m+\frac{h}{2}} \leq \frac{\sqrt{Mm}}{m} + \frac{h\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)+\frac{h}{2}}{\frac{h}{2}} = \sqrt{\frac{M}{m}} + \frac{2\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 3$$

Next, we analyze Situation 2. Because $L_t < 1$ when we order in this situation, we order at most b items. The highest possible price we can be forced to pay for each is M. After replenishing, there are exactly b items in the inventory. The items we order at this time will, on average, be in the inventory for at most a period of $\frac{b+1}{2}$, resulting in an average holding cost of at most $h(\frac{1}{2}b + \frac{1}{2})$. Hence, the average cost, including item price and holding host, is bounded by $M + \frac{h}{2}b + \frac{h}{2}$.

We now argue that for *OPT*, the total item cost for the items used in the period where *UOA* is in Situation 2 is at least $p^* - \frac{h}{2}b$. The intuition of this argument is the same as in Statement 4 in the proof of Theorem 4. Let $\overline{p}$ be

OPT's item cost for the item used at time t in Situation 2, and assume that OPT bought the item at time $t-i$. Note that Lemma 1 holds in this model as well because including an order cost will never make it more attractive to buy items in fractional parts. If $\overline{p} \geq p^* - \frac{h}{2}b$, we are done, so assume to the contrary that $\overline{p} < p^* - \frac{h}{2}b$.

For OPT, we have $\overline{p} = p_{t-i} + ih + \frac{1}{2}h = p_{t-i} + \frac{2i+1}{2}h$. By the assumption that $\overline{p} < p^* - \frac{h}{2}b$, we get that $p_{t-i} < p^* - \frac{h}{2}b - \frac{2i+1}{2}h$. Therefore we have $Q'_{t-i} > 2\frac{p^*-(p^*-\frac{h}{2}b-\frac{2i+1}{2}h)}{h} - 2L_{t-i} + 1 = b + 2(i+1-L_{t-i}) > b$, because $L_t < 1$ implies $L_{t-i} < i+1$. However, if this is the case, UOA would have bought Q'_{t-i} items at time $t-i$. Hence, the inventory level for UOA at time t would be $L_t > L_{t-i} + Q'_{t-i} - i > L_{t-i} + b + 2(i+1-L_{t-i}) - i = i - L_{t-i} + 2 + b \geq 1 + b$, contradicting the fact that in Situation 2, the inventory level is at most b.

This gives us the following ratio with respect to item cost:

$$\frac{M+\frac{h}{2}b+\frac{h}{2}}{p^*-\frac{h}{2}b} = \frac{M+\frac{h}{2}\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)+\frac{h}{2}}{\sqrt{Mm}+h\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)-\frac{h}{2}\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)} = \frac{M+\frac{h}{2}\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)+\frac{h}{2}}{\sqrt{Mm}+\frac{h}{2}\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)}$$

$$\leq \frac{M}{\sqrt{Mm}} + \frac{\frac{h}{2}\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1\right)+\frac{h}{2}}{\frac{h}{2}\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1} = \sqrt{\frac{M}{m}} + 1 + \frac{1}{\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}+1} \leq \sqrt{\frac{M}{m}} + 2$$

In analyzing the order cost, we first argue that OPT never orders more than

$$\left\lceil \frac{M-m}{h} + \sqrt{\frac{S}{h}}\frac{3}{\sqrt{2}} \right\rceil$$

items. This bound is found by considering the extreme cases of prices jumping between m and M. If time were continuous and prices were constant, OPT would order according to the Economic Order Quantity of $\sqrt{\frac{2S}{h}}$. If the price is minimal and increases to maximal, OPT will buy immediately before the increase. If the price stays high after the increase, OPT will order up to the point where the marginal cost (MC) for the items ordered immediately before the increase equals the average cost (AC) after the increase. The marginal cost is the change in total cost (TC) if one more item is bought and can therefore be found by differentiation.

The total cost of placing an order for Q items at the price p is $TC_p(Q) = pQ + S + \frac{1}{2}hQ^2$. Hence, the marginal cost is $MC_p(Q) = p + hQ$ and $AC_p(Q) = p + \frac{S}{Q} + \frac{1}{2}hQ$. So Q is determined by setting $MC_m(Q)$ equal to $AC_M\left(\sqrt{\frac{2S}{h}}\right)$, i.e., $m + hQ = M + \frac{S}{\sqrt{\frac{2S}{h}}} + \frac{1}{2}h\sqrt{\frac{2S}{h}}$, giving $Q = \frac{M-m}{h} + \frac{S}{h\sqrt{\frac{2S}{h}}} + \frac{1}{2}\sqrt{\frac{2S}{h}} = \frac{M-m}{h} + \sqrt{\frac{2S}{h}}$.

Finally, if the price increases from m to M, but decreases to m again, it can be advantages to buy slightly more items than under the assumptions in the above analysis, if this means that one order cost can be avoided because inventory is sufficient to last to the beginning of the m-price period. To determine the number of extra items OPT is willing to buy before the price increase, we compare the extra cost incurred if the items added to the current order to the cost if the items are bought as a separate high price order.

Let this number of items be Q'. We have $Q'm + hQQ' = MQ' + S$, which gives us $Q' = \frac{S}{m-M+hQ} = \frac{S}{m-M+h\left(\frac{M-m}{h}+\sqrt{\frac{2S}{h}}\right)} = \sqrt{\frac{S}{2h}}$. We can conclude that an upper bound on the total number of items that OPT will order at any time is $Q + Q' = \frac{M-m}{h} + \sqrt{\frac{2S}{h}} + \sqrt{\frac{S}{2h}} = \frac{M-m}{h} + \sqrt{\frac{S}{h}}\frac{3}{\sqrt{2}}$. The ceiling is due to the fact that the model works with discrete time.

UOA will buy at least b items at a time in Situation 1 and at least $b-1$ in Situation 2, where the latter is the worst case for UOA.

With a time horizon of T time units, we have the following worst case ratio for the ordering cost:

$$\begin{aligned}\frac{UOA_o}{OPT_o} &\leq \frac{T/(b-1)}{T/\left\lceil \frac{M-m}{h}+\sqrt{\frac{S}{h}}\frac{3}{\sqrt{2}}\right\rceil} \leq \frac{\frac{M-m}{h}+\sqrt{\frac{S}{h}}\frac{3}{\sqrt{2}}+1}{b-1} = \frac{\frac{M-m}{h}}{\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}} + \frac{\sqrt{\frac{S}{h}}\frac{3}{\sqrt{2}}+1}{\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}}} \\ &= \sqrt{\frac{M}{m}}\left(\frac{M-m}{h\sqrt{\frac{S}{h}}} + \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{\frac{S}{h}}}\right) = \sqrt{\frac{M}{m}}\left(\frac{M-m+h}{\sqrt{Sh}} + \frac{3}{\sqrt{2}}\right)\end{aligned}$$

□

In the following theorem, the last fraction in the last minimization term approaches one when M tends towards infinity. Note also that for most practical applications, the first minimization term will be significantly smaller than the second. Again the factor $\sqrt{\frac{M}{m}}$ is central in the result.

Theorem 7. The competitive ratio of UOA is at least

$$\min\left\{\sqrt{\frac{M}{2m}}, \sqrt{\frac{S}{2h}}, \sqrt{\frac{M}{m}} \cdot \frac{1+\frac{h\sqrt{\frac{S}{h}}}{2M\sqrt{\frac{M}{m}}}+\frac{h}{2M}}{1+\frac{\sqrt{Sh}}{\sqrt{2Mm}}-\frac{\sqrt{Sh}}{2M}}\right\}$$

Proof Assume that $\frac{\sqrt{\frac{2S}{h}}}{b}$ is integral. The price sequence is as follows. At time $0, b, 2b, \ldots$, the price is M. At any other time, the price is set such that $Q'_t = b-\varepsilon$. UOA will buy b items at a time exactly when the price is M whereas OPT will buy $\sqrt{\frac{2S}{h}}$ items some insignificant time before UOA buys. We analyze item cost and order cost separately. The final bound will be the smaller of the two.

An upper bound on the price that OPT pays for the items can be calculated by setting Q'_t equal to b and L_t to zero. We have $Q'_t = 2\frac{p^*-p_t}{h} - 2L_t + 1 = b$, which gives us

$$p_t = p^* - \frac{3}{2}hb + \frac{h}{2} = \sqrt{Mm} + hb - \frac{3}{2}hb + \frac{h}{2} = \sqrt{Mm} - \frac{1}{2}hb + \frac{h}{2}$$

With respect to average item cost, including both item price and holding cost, we have

$$UOA_i = M + \frac{h}{2}b = M + \frac{h}{2}\left(\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 1\right)$$

and

$$\begin{aligned} OPT_i &= p_t + \tfrac{h}{2}\sqrt{\tfrac{2S}{h}} = \sqrt{Mm} - \tfrac{h}{2}b + \tfrac{h}{2} + \tfrac{h}{2}\sqrt{\tfrac{2S}{h}} \\ &= \sqrt{Mm} - \tfrac{h}{2}\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + \tfrac{h}{2}\sqrt{\tfrac{2S}{h}} = \sqrt{Mm} - \tfrac{1}{2}\frac{\sqrt{Sh}}{\sqrt{\frac{M}{m}}} + \tfrac{1}{\sqrt{2}}\sqrt{Sh} \end{aligned}$$

Hence, with respect to item cost we have the ratio

$$\frac{UOA_i}{OPT_i} = \frac{M + \frac{h}{2}\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + \frac{h}{2}}{\sqrt{Mm} - \frac{1}{2}\frac{\sqrt{Sh}}{\sqrt{\frac{M}{m}}} + \frac{1}{\sqrt{2}}\sqrt{Sh}} = \frac{\sqrt{\frac{M}{m}} + \frac{h\sqrt{\frac{S}{h}}}{2M} + \frac{h}{2\sqrt{Mm}}}{1 - \frac{\sqrt{Sh}}{2M} + \frac{\sqrt{Sh}}{\sqrt{2Mm}}} = \sqrt{\frac{M}{m}}\,\frac{1 + \frac{h\sqrt{\frac{S}{h}}}{2M\sqrt{\frac{M}{m}}} + \frac{h}{2M}}{1 + \frac{\sqrt{Sh}}{\sqrt{2Mm}} - \frac{\sqrt{Sh}}{2M}}$$

Because of the price sequence, UOA buys b items at a time, whereas OPT will buy at least the amount determined by the economic order quantity of $\sqrt{\frac{2S}{h}}$.
Thus, we have the following ratio with respect to order cost per item:

$$\frac{UOA_o}{OPT_o} \leq \frac{S\sqrt{\frac{2S}{h}}}{Sb} = \frac{\sqrt{\frac{2S}{h}}}{\frac{\sqrt{\frac{S}{h}}}{\sqrt{\frac{M}{m}}} + 1} = \frac{\sqrt{\frac{2S}{h}}}{\frac{\sqrt{\frac{S}{h}} + \sqrt{\frac{M}{m}}}{\sqrt{\frac{M}{m}}}} = \frac{\sqrt{\frac{2S}{h}}\sqrt{\frac{M}{m}}}{\sqrt{\frac{S}{h}} + \sqrt{\frac{M}{m}}} = \frac{\sqrt{\frac{2SM}{hm}}}{\sqrt{\frac{S}{h}} + \sqrt{\frac{M}{m}}}$$

If $\sqrt{\frac{S}{h}} \geq \sqrt{\frac{M}{m}}$, we have

$$\frac{\sqrt{\frac{2SM}{hm}}}{\sqrt{\frac{S}{h}} + \sqrt{\frac{M}{m}}} \geq \frac{\sqrt{\frac{2SM}{hm}}}{2\sqrt{\frac{S}{h}}} = \sqrt{\frac{M}{2m}}$$

On the other hand, if $\sqrt{\frac{S}{h}} < \sqrt{\frac{M}{m}}$, we have

$$\frac{\sqrt{\frac{2SM}{hm}}}{\sqrt{\frac{S}{h}} + \sqrt{\frac{M}{m}}} \geq \frac{\sqrt{\frac{2SM}{hm}}}{2\sqrt{\frac{M}{m}}} = \sqrt{\frac{S}{2h}}$$

□

8. Concluding Remarks

This is a first attempt at establishing results for an online version of the inventory problem. We have obtained tight results on the competitive ratio of online algorithms for the simpler model whereas the gap between the established upper and lower bounds of the algorithms grows with the complexity of the models.

In analyzing the algorithms involving order cost, because of the complexity of the calculations, we are bounding the competitive ratios for order and item costs separately and then balancing to obtain the best possible result. If it were possible to perform a combined analysis, then one might be able to analyze intuitively better algorithms that more directly involve the order cost. An intuitively nice feature of such an algorithm would be that the algorithm and established bounds would tend towards the algorithm and bounds for the model without order cost. In Figure 1, this corresponds to results in the lower part tending towards the results in the upper part as S tends to zero. We leave the design and analysis of such possibly improved algorithms as an interesting open problem.

We have chosen to analyze models with continuous demand, but where replenishment occurs at discrete times. We leave it for further research to investigate the similar problem in continuous review models and in completely discrete models.

Additionally, competitive analysis of online algorithms for problems allowing backorder and price dependency on previous prices would be interesting.

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