DM582 Solutions

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This document contains written solution to exercise problems from the course DM582 (spring 2024). The solutions given here might differ from the solutions discussed in class. In class, we place more emphasis on the intuition leading to the correct answer. Please do not consider reading these solutions an alternative to attending the exercise classes.

References to CLRS refer to the book *Introduction to Algorithms, 4th edition* by Cormen, Leiserson, Rivest, and Stein.

References to KT refer to the book *Algorithm Design, 1st edition* by J. Kleinberg and E. Tardos.

References to Rosen refer to the book *Discrete Mathematics and its Applications, 8th edition* by K. Rosen.

This document will inevitably contain mistakes. If you find some, please report them to me (Mads) so that I can correct them.

Sheet 10

Rosen, 7.2, 6

Exercise

What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?

a) 1 precedes 3.

b) 3 precedes 1.

c) 3 precedes 1 and 3 precedes 2.

Suggested solution

a) The sample space is small, so explicit enumeration of the outcomes of the event is feasible. The permutations for which 1 precedes 3 are 123, 132, and 213. Since the probability of each outcome is $\frac{1}{3!}$, the probability of the event is $\frac{3}{3!} = \frac{1}{2}$.

b) We could enumerate the outcomes again. Alternatively, we can observe that the event that 3 precedes 1 is the complement of the event that 1 precedes 3, so by part a), the probability is $1 - \frac{1}{2} = \frac{1}{2}$.

c) The only permutations for which 3 precedes both 1 and 2 are 312 and 321. Thus, the probability of the event is $\frac{2}{3!} = \frac{1}{3}$.

Rosen, 7.2, 11

Exercise

Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

Suggested solution

Since $E \subseteq E \cup F$ and by the definition of the probability of an event

$$0.7 = p(E) = \sum_{s \in E} p(s) \leq \sum_{s \in E \cup F} p(s) = p(E \cup F).$$

For the second part, we have

$$p(E \cup F) = p(E) + p(F) - p(E \cap F) = 0.7 + 0.5 - p(E \cap F) = 1.2 - p(E \cap F)$$

and since $p(E \cup F) \leq 1$, necessarily $p(E \cap F) \geq 0.2$.

Rosen, 7.2, 36

Exercise

Use mathematical induction to prove that if $E_1, E_2, \ldots, E_n$ is a sequence of n pairwise disjoint events in a sample space S, where n is a positive integer, then $p(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i)$.

Suggested solution

For $n = 1$, the claim is that $p(E_1) = p(E_1)$, which is true. Let $n > 1$. Let $E = E_1 \cup E_2 \cup \cdots \cup E_{n-1}$. By the induction hypothesis, $p(E) = \sum_{i=1}^{n-1} p(E_i)$. We must have $E \cap E_n = \emptyset$ since if there exists some $s \in E \cap E_n$ then $s \in E_i$ for some $i < n$, contradicting that E_n and E_i are disjoint. Thus, by Theorem 2 of section 7.1.3, we have

$$\begin{aligned} kXp(E \cup E_n) &= p(E) + p(E_n) - p(E \cap E_n) \\ &= p(E) + p(E_n) \\ &= \sum_{i=1}^{n-1} p(E_i) + p(E_n) \\ &= \sum_{i=1}^{n} p(E_i) \end{aligned}$$

as desired.

Rosen, 7.2, 38

Exercise

A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

a. What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

b. Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?

Suggested solution

a.) Let E be the event that the sum of the numbers that came up on the two dice is seven, and let F be the event that at least one die came up six. By the definition of conditional probability, we have

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

We start by computing $p(F)$. The event F is the complement of the that at no die came up six, which is $\left(\frac{5}{6}\right)^2$ assuming the dice are rolled independently. Thus, $p(F) = 1 - \left(\frac{5}{6}\right)^2 = 11/36$. There are only two outcomes in the event $E \cap F$, namely $(1, 6)$ and $(6, 1)$, so $p(E \cap F) = \frac{2}{36}$. Thus,

$$p(E \mid F) = \frac{2/36}{11/36} = \frac{2}{11}.$$

Rosen, 7.4, 4

Exercise

A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times?

Suggested solution

For $i = 1, 2 \ldots, 10$, let X_i be an indicator random variable such that $X_i(s) = 1$ if the i-th flip in the outcome s is a head, and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Then $X(s)$ is the number of heads in the outcome s. For each $i = 1, 2, \ldots, 10$ the probability that $X_i = 1$ is 0.6, so $\mathrm{E}(X_i) = 0.6$. By linearity of expectation, we have

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathrm{E}[X_i] = 10 \cdot 0.6 = 6.$$

Rosen, 7.4, 8

Exercise

What is the expected sum of the numbers that appear when three fair dice are rolled?

Suggested solution

For $i = 1, 2, 3$, let X_i be a random variable such that $X_i(s)$ is the value of the i-th die in the outcome s. Let $X = \sum_{i=1}^{3} X_i$. Then $X(s)$ is the sum of the values of the dice in the outcome s. For each $i = 1, 2, 3$, the expected value of X_i is $\mathrm{E}(X_i) = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$. By linearity of expectation, we have

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{3} X_i\right] = \sum_{i=1}^{3} \mathrm{E}[X_i] = 3 \cdot 3.5 = 10.5.$$

Rosen, 7.4, 18

Exercise

Suppose that X and Y are random variables and that X and Y are nonnegative for all points in a sample space S. Let Z be the random variable defined by $Z(s) = max(X(s), Y(s))$ for all elements $s \in S$. Show that $\mathrm{E}(Z) \leq \mathrm{E}(X) + \mathrm{E}(Y)$.

Suggested solution

For any outcome $s \in S$, we have $Z(s) = \max(X(s), Y(s)) \leq X(s) + Y(s)$. Note that the inequality does not necessarily hold if X and Y could be negative. Thus, by the definition of the expected value of a random variable, we have

$$\begin{aligned}
\mathrm{E}[Z] &= \sum_{s \in S} Z(s)p(s) \\
&\leq \sum_{s \in S} (X(s) + Y(s))p(s) \\
&= \sum_{s \in S} X(s)p(s) + \sum_{s \in S} Y(s)p(s) \\
&= \mathrm{E}[X] + \mathrm{E}[Y].
\end{aligned}$$

Note: A broader point here is that if an inequality between random variables holds for all outcomes, then the inequality also holds in expectation.

Rosen, 7.4, 29.a

Exercise

Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

a) What is the expected value of X_n?

Suggested solution

For $i = 1, 2, \ldots, n$, let X_i be a random variable such that $X_i(s) = 1$ if the i-th coin flip in the outcome s is a tail, and -1 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Then $X(s)$ is the number of tails minus the number of heads in the outcome s. For each $i = 1, 2, \ldots, n$, the expected value of X_i is $\mathrm{E}[X_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$. By linearity of expectation, we have

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i] = n \cdot 0 = 0.$$

Rosen, 7.4, 37

Exercise

Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Show that $p(X(s) \geq a) \leq E(X)/a$ for every positive real number a. This inequality is called **Markov's inequality**.

Suggested solution

Recall that, formally, $X(s) \geq a$ is the event $E_{\geq a} = \{s \in S \mid X(s) \geq a\}$. By definition,

$$E[X] = \sum_{s \in S} X(s)p(s).$$

Let $E_{<a} = \{s \in S \mid X(s) < a\}$. Then $(E_{\geq a}, E_{<a})$ is a partition of S into two disjoint sets, so we can split the sum in the above equation into two sums and still have the same terms. Now, we get

$$\begin{aligned} E[X] &= \sum_{s \in E_{\geq a}} X(s)p(s) + \sum_{s \in E_{<a}} X(s)p(s) \\ &\geq \sum_{s \in E_{\geq a}} ap(s) + \sum_{s \in E_{<a}} 0 \cdot p(s) \\ &= a \sum_{s \in E_{\geq a}} p(s) = ap(E_{\geq a}). \end{aligned}$$

Dividing both sides by a gives

$$p(X \geq a) = p(E_{\geq a}) \leq \frac{E[X]}{a}$$

as desired.

Rosen, 7.4, 49

Exercise

What is the expected number of bins that remain empty when m balls are distributed into n bins uniformly at random?

Suggested solution

For $i = 1, 2, \ldots, n$, let X_i be a random variable such that $X_i(s) = 1$ if the i-th bin is empty in the outcome s, and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Then $X(s)$ is the number of bins that are empty in the outcome s. For each $i = 1, 2, \ldots, n$, the probability that $X_i = 1$ is the probability that, for all m balls, the ball is not placed in the i-th bin, which is $\left(\frac{n-1}{n}\right)^m$. Thus, $\mathrm{E}[X_i] = \left(\frac{n-1}{n}\right)^m$. By linearity of expectation, we have

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i] = n \left(\frac{n-1}{n}\right)^m .$$

Rosen, 7.4, supplementary exercise 13

Exercise

Suppose n people, $n \geq 3$, play "odd person out" to decide who will buy the next round of refreshments. The n people each flip a fair coin simultaneously. If all the coins but one come up the same, the person whose coin comes up different buys the refreshments. Otherwise, the people flip the coins again and continue until just one coin comes up different from all the others.

a) What is the probability that the odd person out is decided in just one coin flip?

b) What is the probability that the odd person out is decided with the kth flip?

c) What is the expected number of flips needed to decide the odd person out with n people?

Suggested solution

a) There are two ways for the odd person out to be decided in just one coin flip: either exactly one person flips heads or exactly one person flips tails.

   There are n outcomes in which exactly one person flips heads, n outcomes in which exactly one person flips tails, and none in which both occur since $n \geq 3$. Since all outcomes are equally likely, the probability that the odd person out is decided in just one coin flip is $\frac{2n}{2^n} = \frac{n}{2^{n-1}}$.

b) In order for the odd person out to be decided with the kth flip, the odd person out must not have been decided in the first $k-1$ flips and the kth flip must decide the odd person out. The probability that the odd person out is decided in any given flip is $\frac{n}{2^{n-1}}$ by part a), and thus the probability that the odd person out is not decided in any given flip is $1 - \frac{n}{2^{n-1}}$. From the description of the game, it is clear that each trial is independent. Thus, the probability that the odd person out is decided with the kth flip is

$$\left(1 - \frac{n}{2^{n-1}}\right)^{k-1} \cdot \frac{n}{2^{n-1}}.$$

c) Let X be a random variable such that $X(s) = s$ is the number of flips performed in the outcome s. Then the expected number of flips needed to decide the odd person out is $\mathrm{E}[X]$. We notice that X is a geometric random variable with parameter $p = \frac{n}{2^{n-1}}$. The expected value of a geometric random variable with parameter p is $\frac{1}{p}$, so $\mathrm{E}[X] = \frac{1}{p} = \frac{2^{n-1}}{n}$.

Note: The explanation for the expected value of a geometric random variable can be found in Rosen section 7.4.5. The last step in their derivation requires calculus, and we will not go through it here.

Exercise from course webpage

Exercise

For a graph, $G = (V, E)$, a *spanning bipartite subgraph* G' of G is defined by a partition (V_1, V_2) of V, and the edges $E' \subseteq E$ that have an endpoint in both parts.

Consider the following randomized algorithm for finding a spanning bipartite subgraph of an arbitrary graph: Independently, for each vertex $v \in V$, decide uniformly at random if vertex v is in V_1 or V_2.

1. Give a lower bound on the expected number of edges m' in E' as a function of $m = |E|$.
2. How can you use your result to conclude that any graph has a spanning bipartite subgraph with $m' \geq m/2$?
3. Design a deterministic, polynomial-time algorithm for this problem, finding a spanning bipartite subgraph G' of any graph G, where $m' \geq m/2$.

Suggested solution

1. For $e \in E$ let X_e be an indicator random variable for the event that $e \in |E'|$. Then $X = \sum_{e \in E} X_e$ is a random variable whose value is the number of edges m' in the resulting bipartite graph. An edge $uv \in E'$ iff u and v are in different parts of the partition (V_1, V_2), which happens with probability $\frac{1}{2}$. Thus,

$$E[X_e] = \frac{1}{2}$$

for any $e \in E$ and by linearity of expectation

$$E[X] = \sum_{e \in E} \frac{1}{2} = \frac{1}{2} m.$$

2. Since the expected number of edges in a bipartite graph obtained from a random partition is $\frac{1}{2}m$, there must be some partition (V_1, V_2) such that the number of edges in the induced bipartite graph is at least $\frac{1}{2}m$.

3. The following algorithm achieves this:

(i) Initially, set $V_1 = V$ and $V_2 = \emptyset$.

(ii) Repeatedly pick $v \in V_i$ for some $i \in [2]$ such that v has more than half its neighbors in V_i and move v to V_{3-i}.

(iii) When no such vertex v exists, return (V_1, V_2).

We observe that when moving a vertex v from V_i to V_{3-i}, the number of edges in G' only increases by the choice of v. Thus, the given procedure is indeed an algorithm (it terminates).

When the algorithm terminates, $d_{G'}(v) \geq \frac{1}{2} d_G(v)$ for all $v \in V$ and thus

$$\begin{aligned} m' &= \frac{1}{2} \sum_{v \in V} d_{G'}(v) \\ &\geq \frac{1}{2} \sum_{v \in V} \frac{1}{2} d_G(v) \\ &= \frac{1}{4} \sum_{v \in V} d_G(v) \\ &= \frac{1}{2} m \end{aligned}$$

where we use that $\sum_{v \in V(H)} d_H(v) = 2|E(H)|$ for any graph H.[1]

[1]For a graph H, $d_H(v)$ denotes the degree of the vertex v in H