Towards Automation of Real Analysis in Coq

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We have finished a constructive formalization in the theorem prover Coq of the Fundamental Theorem of Calculus, which states that differentiation and integration are inverse processes. This formalization is built upon the library of constructive algebra created in the FTA (Fundamental Theorem of Algebra) project, which is extended with results about the real numbers, namely about (power) series.

This formalization was done closely following the work of Bishop [1]; the real numbers were first axiomatically characterized as a complete ordered field with the archimedian property; later, this axiomatization was proved by Geuvers and Niqui [4] to be appropriate (in the sense that the construction of real numbers as Cauchy sequences of rationals satisfies the axioms) and categorical (as any two models of these axioms are isomorphic).

Using this work as a basis, partial functions are defined as a Coq record type consisting of a predicate and a total function on the set of real numbers that satisfy that predicate (see [3]). The usual operations (composition, addition, multiplication, division) are then defined as yielding partial functions from partial functions. We can then define continuity, differentiability and integration, and prove the usual properties of these: preservation of continuity and differentiability through algebraic operations and functional

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composition, uniqueness of derivative and the rules for computing derivatives. Using these, we can then formalize Bishop's proofs of the constructive versions of Rolle's theorem, the Mean Law and Taylor's theorem, as well as of the Fundamental Theorem of Calculus.

One of the most important issues throughout this work is automating routine tasks, such as computing the derivative of a function. This was successfully done in Coq by using the general method of reflection as described in [7] and [5] adapted to our domain to define new tactics suitable for use with specific kinds of goals. With these tactics further development of the theory became much more high-level, allowing proofs to be done at a level of detail more similar to what is usual in mathematics.

Finally, the usual elementary transcendental functions (exponential, sinus, cosinus, tangent and their inverses) were defined as examples of partial functions and their properties were proved using the theoretical tools previously formalized.

Future work will include developing a higher level of automation, including the use of reflection to build a new tactic that can automatically prove a large class of equalities of real numbers.

References

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