Heterogeneous fibring of deductive systems via abstract proof systems

Luís Cruz-Filipe, *SQIG-IT and Dep. Informatics, FC, U Lisbon*

Amílcar Sernadas, *SQIG-IT and Dep. Mathematics, IST, TU Lisbon*

Cristina Sernadas, *SQIG-IT and Dep. Mathematics, IST, TU Lisbon*

Abstract

Fibring is a meta-logical constructor that applied to two logics produces a new logic whose formulas allow the mixing of symbols. Homogeneous fibring assumes that the original logics are presented in the same way (e.g via Hilbert calculi). Heterogeneous fibring, allowing the original logics to have different presentations (e.g. one presented by a Hilbert calculus and the other by a sequent calculus), has been an open problem. Herein, consequence systems are shown to be a good solution for heterogeneous fibring when one of the logics is presented in a semantic way and the other by a calculus and also a solution for the heterogeneous fibring of calculi. The new notion of abstract proof system is shown to provide a better solution to heterogeneous fibring of calculi namely because derivations in the fibring keep the constructive nature of derivations in the original logics. Preservation of compactness and semi-decidability is investigated.

Keywords: heterogeneous fibring, abstract proof system, preservation

1 Introduction

A mechanism for combining logics is an operation on a (sub)class of logics in the sense that it provides the means for obtaining a new logic from a finite number of logics (for instance, fusion is an operation on the subclass of modal logics while fibring is an operation on the class of logics). For a nice and gentle motivation of the topic see [3], and for an early example of the combination of tense and modality see [24]. The different methods for combination depend upon and impose different presentations of the original logics ranging, on one hand, from very abstract to more concrete ones and, on the other hand, from deductive-based to semantic-based. In general, it is assumed that the logics to be combined are presented in the same way. For instance, the logics to be combined are endowed with a Hilbert calculus.

In this paper, we address the problem of combining logics presented in different styles (e.g. a Hilbert calculus and a sequent calculus). This is an open problem of great practical interest, since the requirement that the two logics to be combined be presented in the same way is often not met in practice.

A very abstract presentation of a logic is via consequence systems (with no deductive or semantic connotation). A consequence system is a pair composed by a set (of formulas, usually with no details on the construction) and a binary (consequence) relation (the pairs $\langle \Gamma, \varphi \rangle$ indicate that formula φ is a consequence of set of formulas Γ). Some properties are required for this consequence relation. The presentation of logics via consequence systems allows the definition of some simple forms of combining logics like, for example, union of logics (the set of formulas in the union is the union of the sets of formulas of the components and the consequence relation is also the union of the original consequence relations).

More interesting combination mechanisms can be defined when describing logics in a more concrete way. In the case of fibring (see [11, 12]), it is necessary to indicate the signatures (set of symbols) of the (usually two) original logics. The set of formulas is the free algebra generated by the set of symbols in the signature and a set of (schema) variables. A signature of the fibring is the union of original signatures (but the set of formulas is not the union of the set of formulas of the components since in the same formula we can have symbols from both signatures). Signatures are also needed for the fusion of modal logics, see [24], as well as when combining temporal logic systems, see [9].

Also the consequence relation can be more concrete, for instance, when it is generated from a Hilbert calculus (with axioms and rules). The induced consequence relation is then defined in a constructive way. Another possibility for defining the consequence system is via semantics by giving pairs composed by a class of models and a satisfaction (binary) relation (a pair $\langle m, \varphi \rangle$ means that model m satisfies formula φ). In the induced consequence system, the consequence relation is then the semantic entailment.

Fusion of two normal modal logics presented by Hilbert calculi is a bi-modal logic presented by an Hilbert calculus whose axioms and rules are the union of the axioms and rules for both of them. But fusion of modal logics can also be explained in semantic terms. The models of the fusion are bi-Kripke structures with the same set of worlds but different accessibility relations. Similar examples can be given for fibring. For example, the fibring of logics presented by Hilbert calculi is a logic presented by a Hilbert calculus having the axioms and the rules of the component Hilbert calculi. Also some results on the fibring of tableau systems can be found in [8, 1].

In the examples above, we are implicitly thinking of combining logics in a homogeneous scenario: both logics are presented in the same way either by Hilbert calculi or by Kripke structures. However, this is not usually the case. Heterogeneous combination of logics, in particular heterogeneous fibring, is an open problem identified by Gabbay in [11]. That is, we would like to be able to combine logics presented in a different way, for instance, to define the combination of two modal logics, one deductively presented by a Hilbert calculus and the other semantically presented by Kripke structures. Note that heterogeneous combination was never dealt with any of the existing combination mechanisms.

The heterogeneous scenario was dealt with in [21] for fibring logics presented semantically but with different semantic domains, for instance fibring modal logic presented by Kripke structures with first-order logic with the usual first-order structures. The solution was to introduce an algebraic structure as the main semantic primitive and to define fibring of such algebraic structures. Hence when we are given a logic presented semantically, the first step is to determine how to extract an algebraic structure from each model. A criterion for the correction of the extraction mechanism is to prove that the entailment of the original logics is preserved.

However, so far, there was no solution for the problems of combining two logics when: (i) one is presented in a semantic way and the other is presented by a calculus (Hilbert, sequent, tableau, etc); (ii) both are presented by calculi but these are of different kinds, say one Hilbert and the other sequent. This situation may arise in practice: an example would be the study of the behavior through time of a system whose state logic is presented via a Hilbert calculus and where the temporal logic is given as a sequent calculus. Applying the current-day fibring techniques would require changing the presentation of one (or both) of the logics, which is neither easy nor convenient. This is the problem we address in this paper.

A very important issue in combination of logics is preservation of properties. That is, assuming that the original logics have a given property (say decidability), we want to know whether the combination still has this property (of being decidable). In several cases, preservation holds provided that we restrict the logics at hand: that is, only when we work in a particular subclass of the class of logics.

Among the methods for combining logics, fusion of modalities (logics presented by Hilbert calculi at the deductive level and Kripke structures at the semantic level) [24] is the best understood in what concerns preservation of properties as soundness, weak completeness, uniform Craig interpolation (for theoremhood) and decidability via finite model property (see [26, 19, 13]). Further results on preservation of weak completeness can be seen in [10]. Preservation results were also obtained in the context of temporalization (adding a temporal dimension to an original logic) as in [27, 18]. Some interesting preservation properties can also be found for the product of (modal) logics, see [13, 14, 15, 16].

Fibring is a more general mechanism since it goes beyond modal logic. For instance, we can think of fibring relevance logic with intuitionistic logic, or modal logic and first-order logic. Although preservation of soundness, completeness and interpolation has been already investigated in the context of propositional-based logics [28, 23, 5], first-order quantification [22], higher-order quantification [7], non truth-functional semantics [4], sequent calculus and other deductive systems [17, 20], other forms of preservation are still to be fully understood, namely the ones related to compactness, decidability and complexity.

The main objective of this paper is to provide solutions to open problems in heterogeneous fibring. The solution to (i), which is also a solution to (ii), is to define fibring of consequence systems (observe that our notion of consequence system differs slightly from the usual one because we need to include the signature as a component). However, the constructive nature of derivations is lost. Hence we provide another solution to (ii) introducing the new concept of abstract proof system. We define fibring of proof systems and keep, in the fibring, the constructive nature of derivations. Examples are provided for modal logic. Preservation of compactness, semi-decidability and effectiveness is investigated.

The paper is organized as follows. Section 2 concentrates on consequence systems as a possible solution to heterogeneous fibring. Fibring of consequence systems is defined as a (Tarski) fixed point operator. Section 3 focuses on the new notion of (abstract) proof system. We introduce the proof systems induced by Hilbert, sequent and tableau calculi and define fibring of proof systems. In both settings several preservation results are proved. Section 4 introduces some auxiliary notions for those not so familiar with computability issues.

2 Consequence systems

We start by defining consequence system and identifying several classes of consequence systems (e.g. closed for substitution, compact or finitary and (strongly) semi-decidable). We prove some results about the relationship between these classes. Then we show how different kinds of calculi generate consequence systems. The last subsection is dedicated to fibring consequence systems.

When discussing decidability results, we will rely heavily on the Church–Markov–Turing postulate and work with the intuitive concepts.

2.1 Basics

We are interested in dealing with propositional-based logics in order to investigate the issue of heterogeneous fibring in a simple context.

A *signature* C is a family of sets indexed by the natural numbers. The elements of each C_k are called *constructors or connectives* of arity k. We say that $C \subseteq C'$ if $C_k \subseteq C'_k$ for every $k \in \mathbb{N}$.

Let $L(C, \Xi)$ be the free algebra over C generated by $\Xi = \{\xi_n : n \in \mathbb{N}\}$. In the sequel we will write $L(C)$ instead of $L(C, \Xi)$. The elements of $L(C)$ are called *formulas* and $L(C)$ is the language. The elements of Ξ are schema variables that will allow the definition of schematic derivations. A derivation can be obtained from a schematic derivation by using a substitution.

A *substitution* is any map $\sigma : \Xi \to L(C)$. Substitutions can be inductively extended to formulas and to sets of formulas: $\sigma(\gamma)$ is the formula where each $\xi \in \Xi$ is replaced by $\sigma(\xi)$; $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$.

A *consequence system* is a tuple $\langle C, \vdash\rangle$ where C is a signature and $\vdash : \wp L \to \wp L$ is a map with the following properties:

- Extensivity: $\Gamma \subseteq \Gamma^{\vdash}$;
- Monotonicity: If $\Gamma_1 \subseteq \Gamma_2$ then $\Gamma_1^{\vdash} \subseteq \Gamma_2^{\vdash}$;
- Idempotence: $(\Gamma^{\vdash})^{\vdash} \subseteq \Gamma^{\vdash}$;
- Closure for renaming substitutions: $\rho(\Gamma^{\vdash}) \subseteq (\rho(\Gamma))^{\vdash}$ for every renaming substitution ρ (that is, a map such that $\rho(\xi) \in \Xi$ for every $\xi \in \Xi$).

We say that $\Gamma^{\vdash}$ is the closure of Γ. Note that $(\Gamma^{\vdash})^{\vdash} = \Gamma^{\vdash}$ is a trivial consequence of the properties above.

A pair $\langle C, \vdash\rangle$ is a *quasi consequence system* if $\vdash$ has all the above properties with the possible exception of idempotence. A very simple though not so interesting case is the one where $\Gamma^{\vdash} = \Gamma$ for every $\Gamma \subseteq L(C)$. Such a consequence system is a topological (Kuratowski) closure operator since it also satisfies $\emptyset^{\vdash} = \emptyset$ and $(\Gamma_1 \cup \Gamma_2)^{\vdash} = \Gamma_1^{\vdash} \cup \Gamma_2^{\vdash}$ for every $\Gamma_1, \Gamma_2 \subseteq L(C)$. But the really interesting consequence systems do not satisfy empty and union properties although they can enjoy other properties.

A consequence system is said to be *non-trivial* if $\xi \notin \Pi^{\vdash}$ for every $\xi \in \Xi \setminus \Pi$ and $\Pi \subseteq \Xi$. A consequence system is *closed for substitution* if $\sigma(\Gamma^{\vdash}) \subseteq (\sigma(\Gamma))^{\vdash}$ for any substitution σ. A consequence system is *compact or finitary* if $\Gamma^{\vdash} = \bigcup_{\Phi \in \wp_{\text{fin}}\Gamma} \Phi^{\vdash}$ for every $\Gamma \subseteq L(C)$, where $\wp_{\text{fin}}\Gamma$ is the set of all finite subsets of Γ.

A consequence system $\langle C, \vdash\rangle$ is *semi-decidable* if $\Gamma^{\vdash}$ is recursively enumerable[1]

[1] A set X is recursive if there is an algorithm that decides whether $x \in X$ or not; a set X is recursively enumerable

whenever Γ is recursive. A consequence system is *strongly semi-decidable* if $\Gamma^{\vdash}$ is a recursively enumerable set for every recursively enumerable set $\Gamma \subseteq L(C)$. A natural question is whether there is a consequence system where the closure of a recursive set is not always a recursively enumerable set. We provide the following illustration.

EXAMPLE 2.1
Consider the consequence system $\langle \mathbb{N}, \vdash \rangle$ such that $\Gamma^{\vdash} = \Gamma \cup A$ where A is any non recursively enumerable set. Then $\emptyset^{\vdash}$ is not a recursively enumerable set even though $\emptyset$ is recursive. $\triangleleft$

The relationship between semi-decidable and strongly semi-decidable consequence systems can be investigated. We observe that a strongly semi-decidable consequence system is semi-decidable, which is a consequence of the fact that every recursive set is a recursively enumerable set.

PROPOSITION 2.2
A compact and semi-decidable consequence system is strongly semi-decidable.

PROOF. Let $\mathcal{C}$ be a compact and semi-decidable consequence system
and $\Gamma \subseteq L(C)$ be a recursively enumerable set. Then either Γ is the empty set or there is a total recursive function $f : \mathbb{N} \to L(C)$ such that $f(\mathbb{N}) = \Gamma$. If $\Gamma = \emptyset$ then Γ is recursive, and so by hypothesis $\Gamma^{\vdash}$ is recursively enumerable. In the other case consider an enumeration $\{\Gamma_n : n \in \mathbb{N}\}$ of $\wp_{\text{fin}}\Gamma$. (For example, given $n \in \mathbb{N}$, write n as a sum of powers of two

$$n = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} a_i 2^i$$

and let $\Gamma_n = \{f(i) : a_i = 1\}$. If $\Phi = \{\varphi_0, \ldots, \varphi_m\} \subseteq \Gamma$, then there are numbers $i_0, \ldots, i_m$ such that $f(i_k) = \varphi_k$, and then

$$\Gamma_{\sum_{k=0}^{m} 2^{i_k}} = \Phi.)$$

For all n, Γ_n is recursive (since it is finite), so $\Gamma_n^{\vdash}$ is a recursively enumerable set. Therefore,

$$\bigcup_{n \in \mathbb{N}} \Gamma_n^{\vdash}$$

is a recursively enumerable set (it is a recursive union of recursively enumerable sets), and since $\mathcal{C}$ is compact this set coincides with $\Gamma^{\vdash}$. $\blacksquare$

Consequence systems can be related. We say that consequence system $\langle C, \vdash \rangle$ is *weaker* than consequence system $\langle C', \vdash' \rangle$, indicated by

$$\langle C, \vdash \rangle \leq \langle C', \vdash' \rangle,$$

if $C \subseteq C'$ and $\Gamma^{\vdash} \subseteq \Gamma^{\vdash'}$ for every $\Gamma \subseteq L(C)$. The relation $\leq$ introduced in the class of consequence systems is reflexive, transitive and anti-symmetric.

---

if there is a procedure that answers affirmatively to the question of whether $x \in X$ whenever that is the case, but may never answer otherwise.

2.2 Induced consequence systems

We now show that different kinds of (general) calculi, as well as logics presented via their semantics, induce consequence systems. We stress that the notion of consequence system also covers calculi having infinitary rules like some versions of linear temporal logic where the rule

$$\frac{X^n\xi : n \in \mathbb{N}}{G\xi}$$

is included. This rule means that if ξ holds now ($X^0\xi$) and in each subsequent instant ($X^n\xi$) then it always holds in the future ($G\xi$). Later on we discuss a non-compact situation.

2.2.1 Hilbert calculi

A *Hilbert calculus* is a pair $H = \langle C, R\rangle$ where C is a signature and R is a set of (Hilbert) rules, i.e. pairs $\langle\Theta, \eta\rangle$ with $\Theta \cup \{\eta\} \subseteq L(C)$ and Θ finite. Rules are schematic in the sense that the elements of Ξ can be instantiated. A rule where $\Theta = \emptyset$ is said to be an axiom.

The formula φ is *Hilbert-derived* from the set of formulas Γ, indicated by $\Gamma \vdash_H \varphi$, iff there is a finite sequence (a derivation) $\varphi_1 \dots \varphi_n$ of formulas such that φ_n is φ and for each $i = 1, \dots, n$ one of the following holds.

- φ_i is an element of Γ (justified by Hyp);
- there exist a rule $r = \langle\Theta, \eta\rangle$ and a substitution σ such that $\varphi_i = \sigma(\eta)$ and $\sigma(\Theta) \subseteq \{\varphi_1, \dots, \varphi_{i-1}\}$ (justified by r).

The following result is easy to prove observing that the pair $\langle\sigma(\Theta), \sigma(\eta)\rangle$ is an instance of the rule $\langle\Theta, \eta\rangle$.

PROPOSITION 2.3
A Hilbert calculus H induces a consequence system $\mathcal{C}(H) = \langle C, \vdash_H\rangle$ such that $\Gamma^{\vdash_H} = \{\varphi : \Gamma \vdash_H \varphi\}$. Furthermore, $\mathcal{C}(H)$ is compact and closed for substitution.

EXAMPLE 2.4
A Hilbert calculus $H_B = \langle C, R\rangle$ for modal logic with the B axiom (sound with respect to symmetric frames) is as follows:

- $C_0 = \Pi$, $C_1 = \{\neg, \Box\}$, $C_2 = \{\Rightarrow\}$;
- $R = \{\langle\emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle\emptyset, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))\rangle,$
  $\langle\emptyset, (((\neg\,\xi_1) \Rightarrow (\neg\,\xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
  $\langle\emptyset, ((\Box(\xi_1 \Rightarrow \xi_2)) \Rightarrow ((\Box\xi_1) \Rightarrow (\Box\xi_2)))\rangle,$
  $\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle,$
  $\langle\{\xi_1\}, (\Box\xi_1)\rangle$
  $\langle\emptyset, (\xi_1 \Rightarrow (\Box(\Diamond(\xi_1))))\rangle\}.$

Observe that the rules are schematic: the elements of Ξ can be instantiated by any formulas. The last rule is usually known as the B axiom. All the other rules are the usual ones for normal modal logic K. $\triangleleft$

2.2.2 Sequent calculi

A *sequent* over a signature C is a pair $\langle \Delta_1, \Delta_2 \rangle$, denoted by $\Delta_1 \to \Delta_2$, where Δ_1 (the antecedent) and Δ_2 (the consequent) are multi-sets of formulas in $L(C)$. A (sequent) *rule* is a pair $\langle \{\Theta_1, \dots, \Theta_n\}, \Omega \rangle$, indicated by

$$\frac{\Theta_1 \quad \dots \quad \Theta_n}{\Omega},$$

where $\Theta_1, \dots, \Theta_n$ (the premises) and Ω (the conclusion) are sequents. A *sequent calculus* is a pair $\langle C, R \rangle$, where C is a signature and R is a set of rules including structural rules and specific rules for the connectives.

- Structural rules:

$$\frac{\xi_1, \Delta_1 \to \Delta_2 \quad \Delta_1 \to \Delta_2, \xi_1}{\Delta_1 \to \Delta_2}\ \mathsf{Cut}$$

$$\frac{\Delta_1 \to \Delta_2}{\Delta_1 \to \Delta_2, \xi_1}\ \mathsf{RW} \qquad \frac{\Delta_1 \to \Delta_2}{\xi_1, \Delta_1 \to \Delta_2}\ \mathsf{LW}$$

$$\frac{\Delta_1 \to \xi_1, \xi_1, \Delta_2}{\Delta_1 \to \xi_1, \Delta_2}\ \mathsf{RC} \qquad \frac{\Delta_1, \xi_1, \xi_1 \to \Delta_2}{\Delta_1, \xi_1 \to \Delta_2}\ \mathsf{LC}$$

- Left rules: the conclusion has the form $c(\varphi_1, \dots, \varphi_n), \Delta_1 \to \Delta_2$ for some n-ary connective c.
- Right rules: the conclusion has the form $\Delta_1 \to \Delta_2, c(\varphi_1, \dots, \varphi_n)$ for some n-ary connective c.

A sequent s is *derivable* from a set of sequents H, denoted by $H \vdash_G s$, if there is a finite sequence (a derivation) of sequents $\Delta_{1,1} \to \Delta_{2,1} \dots \Delta_{1,n} \to \Delta_{2,n}$ such that:

- $\Delta_{1,1} \to \Delta_{2,1}$ is s;
- for each $i = 1, \dots, n$, one of the following holds:
  - $\Delta_{1,i} \to \Delta_{2,i}$ is an axiom (justified by Ax), that is $\Delta_{1,i} \cap \Delta_{2,i} \neq \emptyset$;
  - $\Delta_{1,i} \to \Delta_{2,i} \in H$ (justified by Hyp);
  - there exist a rule $r = \langle \{\Theta_1, \dots, \Theta_k\}, \Omega \rangle$ and a substitution σ such that $\Delta_{1,i} \to \Delta_{2,i} = \sigma(\Omega)$ and $\sigma(\Theta_j) \in \{\Delta_{1,i+1} \to \Delta_{2,i+1}, \dots \Delta_{1,n} \to \Delta_{2,n}\}$ for $j = 1, \dots, k$ (justified by r and the indexes of $\sigma(\Theta_j)$).

We say that a formula φ is *sequent-derivable* from the set of formulas Γ, indicated by $\Gamma \vdash_G \varphi$, if $\vdash_G \Gamma \to \varphi$. The following result is easy to prove.

Proposition 2.5
A sequent calculus G induces a consequence system $\mathcal{C}(G) = \langle C, \vdash_G \rangle$ such that $\Gamma^{\vdash_G} = \{\varphi : \Gamma \vdash_G \varphi\}$. Furthermore, $\mathcal{C}(G)$ is compact and closed for substitution.

Example 2.6
A sequent calculus G_{S4} for modal logic S4 (characterized by reflexive and transitive frames) has the following specific rules, as presented in page 287 of [25], where $(\Box\Delta)$ is $\{(\Box\delta) : \delta \in \Delta\}$ and $(\Diamond\Delta)$ is $\{(\Diamond\delta) : \delta \in \Delta\}$:

$$\frac{\Delta_1, \xi_1 \to \xi_2, \Delta_2}{\Delta_1 \to (\xi_1 \Rightarrow \xi_2), \Delta_2}\ R\Rightarrow \qquad \frac{\Delta_1 \to \xi_1, \Delta_2 \quad \Delta_1, \xi_2 \to \Delta_2}{\Delta_1, (\xi_1 \Rightarrow \xi_2) \to \Delta_2}\ L\Rightarrow$$

$$\frac{\Delta_1, \xi_1 \to \Delta_2}{\Delta_1 \to (\neg\, \xi_1), \Delta_2}\ R\neg \qquad \frac{\Delta_1 \to \xi_1, \Delta_2}{\Delta_1, (\neg\, \xi_1) \to \Delta_2}\ L\neg$$

$$\frac{(\Box\Delta_1) \to \xi_1, (\Diamond\Delta_2)}{\Delta_1', (\Box\Delta_1) \to (\Box\xi_1), (\Diamond\Delta_2), \Delta_2'}\ R\Box \qquad \frac{\Delta_1, \xi_1, (\Box\xi_1) \to \Delta_2}{\Delta_1, (\Box\xi_1) \to \Delta_2}\ L\Box$$

Observe that weakening and contraction can be derived from these rules. As an example, we can derive $\{\to \xi_1\} \vdash_{G_{S4}} \ \to (\Box\xi_1)$. *Note that we can extract a sequent calculus for propositional logic by eliminating rules* $R\Box$ *and* $L\Box$. *It is also easy to get the right and left rules for* $\Diamond$ *using the abbreviation* $(\Diamond\varphi)$ *is* $(\neg(\Box(\neg\,\varphi)))$. ◁

2.2.3 Tableau calculi

Most of the tableau calculi rely on the existence of a negation in the logic at hand. To be able to deal with as many logics as possible we avoid this assumption by considering tableau calculi over labelled formulas. Herein we only consider a very simple case for the labels. A *labelled formula* is a pair $\langle\varphi, i\rangle$, indicated by $i\!:\!\varphi$, where i is either 0 or 1. Intuitively speaking, $1\!:\!\varphi$ states that we want φ to be true and $0\!:\!\varphi$ means that we want φ to be false. We denote by L^λ the set of pairs $i:\varphi$ such that $i = 0, 1$ and $\varphi \in L(C)$. A (tableau) *rule* is a pair $\langle\Upsilon, \mu\rangle$ where $\Upsilon \in \wp_{\mathrm{fin}}\wp_{\mathrm{fin}}L^\lambda(C)$ and $\mu \in L^\lambda(C)$. The formula μ is the conclusion of the rule and each set in Υ is said to be an alternative. We can look at Υ as alternatives to μ. A *tableau calculus* is a pair $\langle C, R\rangle$ where C is a signature and R is a set including the following rules:

- EM (excluded middle): $\langle\{\{\varphi, 1\!:\!\xi\}, \{\varphi, 0\!:\!\xi\}\}, \varphi\rangle$;
- for each connective c, a positive rule with conclusion $1\!:\!c(\varphi_1, \ldots, \varphi_k)$ and a negative rule with conclusion $0\!:\!c(\varphi_1, \ldots, \varphi_k)$.

Rule EM is not always included in the definition of a tableau calculus, but it can easily be shown to be admissible (see [2] for details) if the tableau system is known to be complete, otherwise the proof of admissibility is not straightforward (coinciding with cut-elimination in sequent calculi). We chose to include it in our definition as it makes the work further on easier.

A *tableau-derivation* of a set of labelled formulas Θ from a set H of sets of labelled formulas is a sequence $\Psi_1 \ldots \Psi_n$ of finite sets of labelled formulas such that:

- Ψ_1 is Θ;
- for each $i = 1, \ldots, n$, one of the following holds:
  - there is a $\psi \in L(C)$ such that $1\!:\!\psi, 0\!:\!\psi \in \Psi_i$ (justified by Abs);
  - $\Psi_i \in H$ (justified by Hyp);
  - there exist a substitution σ and a rule $r = \langle\Upsilon, \mu\rangle$ such that $\sigma(\mu) \in \Psi_i$ and for each $\upsilon \in \Upsilon$, $\sigma(\upsilon) \cup (\Psi_i \setminus \sigma(\mu)) \in \{\Psi_{i+1}, \ldots, \Psi_n\}$ (justified by r).

We say that a formula φ is *tableau-derivable* from the finite set of formulas Δ, indicated by $\Delta \vdash_S \varphi$, if $\vdash_S \{(1:\delta) : \delta \in \Delta\} \cup \{0\!:\!\varphi\}$. We say that φ is derivable from Γ (not necessarily finite), $\Gamma \vdash_S \varphi$, if $\Delta \vdash_S \varphi$ for some finite $\Delta \subseteq \Gamma$. The following result is easy to prove.

Proposition 2.7
A tableau calculus S *induces a consequence system* $\mathcal{C}(S) = \langle C, \vdash_S\rangle$ *such that* $\Gamma^{\vdash_S} = \{\varphi : \Gamma \vdash_S \varphi\}$. *Furthermore,* $\mathcal{C}(S)$ *is compact and closed for substitution.*

Example 2.8
A tableau calculus $S_{P_{\wedge,\Rightarrow}}$ for the propositional connectives $\wedge$ and $\Rightarrow$ has the following specific rules:

$$\frac{\{1\!:\!\xi_1, 1\!:\!\xi_2\}}{1\!:\!(\xi_1 \wedge \xi_2)}\ 1\wedge \qquad \frac{\{0\!:\!\xi_1\} \quad \{0\!:\!\xi_2\}}{0\!:\!(\xi_1 \wedge \xi_2)}\ 0\wedge$$

$$\frac{\{0\!:\!\xi_1\} \quad \{1\!:\!\xi_2\}}{1\!:\!(\xi_1 \Rightarrow \xi_2)}\ 1\Rightarrow \qquad \frac{\{1\!:\!\xi_1, 0\!:\!\xi_2\}}{0\!:\!(\xi_1 \Rightarrow \xi_2)}\ 0\Rightarrow$$

Observe that the rule $0\wedge$ states that there are two alternatives for a conjunction to be false. $\triangleleft$

2.2.4 Interpretation structures

Now we show that semantic structures also induce consequence systems. An *interpretation structure* is a triple $S = \langle C, M, \Vdash\rangle$ where C is a signature, M is a class and $\Vdash \subseteq M \times L(C)$. The elements of the class are called models and $\Vdash$ is the satisfaction relation. We denote by $\mathrm{Mod}(\varphi)$ the set $\{m \in M : m \Vdash \varphi\}$ and by $\mathrm{Mod}(\Gamma) = \bigcap_{\gamma\in\Gamma} \mathrm{Mod}(\gamma)$.

Proposition 2.9
The interpretation structure S induces a system $\mathcal{C}(S) = \langle C, \vDash\rangle$ where $\Gamma^{\vDash} = \{\varphi \in L(C) : Mod(\Gamma) \subseteq Mod(\varphi)\}$ that has all the properties of a consequence system with the possible exception of the closure under renaming substitutions.

An interpretation system S is *sensible-to-renaming* if for each renaming substitution ρ there is a map $\beta_\rho : M \to M$ such that $m \Vdash \rho(\varphi)$ iff $\beta_\rho(m) \Vdash \varphi$.

Proposition 2.10
Let S be a sensible-to-renaming interpretation structure. Then $\mathcal{C}(S) = \langle C, \vDash\rangle$ is a consequence system.

Proof. Let ρ be a renaming substitution and $\varphi \in \Gamma^{\vDash}$. Then $\mathrm{Mod}(\Gamma) \subseteq \mathrm{Mod}(\varphi)$. Assume that $m \in \mathrm{Mod}(\rho(\Gamma))$. Then $\beta_\rho(m) \in \mathrm{Mod}(\Gamma)$, hence $\beta_\rho(m) \in \mathrm{Mod}(\varphi)$ and so $m \in \mathrm{Mod}(\rho(\varphi))$. ∎

Example 2.11
The (Kripke) interpretation system S_B for modal logic with axiom B (sound with respect to symmetric frames) is as follows:

- *C is the same as in Example 2.4;*
- *each model is a tuple (Kripke structure) $\langle W, R, V\rangle$ where W is a non-empty set, $R \subseteq W \times W$ is a binary relation symmetric and transitive and $V : \Xi \to \wp W$ is a map;*
- *$m \Vdash \varphi$ if $m, w \Vdash \varphi$ for every $w \in W$, where:*
  – *$m, w \Vdash \xi$ if $w \in V(\xi)$;*
  – *$m, w \Vdash (\neg\,\varphi)$ if not $m, w \Vdash \varphi$;*
  – *$m, w \Vdash (\varphi_1 \wedge \varphi_2)$ if $m, w \Vdash \varphi_1$ and $m, w \Vdash \varphi_2$;*
  – *$m, w \Vdash (\Box\varphi)$ if $m, u \Vdash \varphi$ for every $u \in W$ such that wRu.*

The induced $\mathcal{C}(S_B)$ is a consequence system. Indeed S_B is sensible-to-renaming. Let ρ be a renaming substitution and let $\beta_\rho(\langle W,R,V\rangle) = \langle W,R,V'\rangle$ where $V' = V\circ\rho$. It is easy to prove by induction on the structure of the formula φ that $\langle W,R,V\rangle, w \Vdash \rho(\varphi)$ iff $\langle W,R,V'\rangle, w \Vdash \varphi$. ◁

2.3 Fibring

The language of the fibring of two consequence systems will be generated by the union of the connectives in both signatures. An essential ingredient for the definition of the consequence relation will be the ability to translate formulas of the fibring to either component. We achieve this by renaming the schema variables and coding formulas by fresh variables.

Assume that $C \subseteq C'$ and let $g : L(C') \to \mathbb{N}$ be a bijection. The *translation*

$$\tau_g : L(C') \to L(C)$$

is a map defined inductively as follows:

- $\tau_g(\xi_i) = \xi_{2i+1}$ for $\xi_i \in \Xi$;
- $\tau_g(c) = c$ for $c \in C_0$;
- $\tau_g(c(\gamma'_1,\ldots,\gamma'_k)) = c(\tau_g(\gamma'_1),\ldots,\tau_g(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1,\ldots,\gamma'_k \in L(C')$;
- $\tau_g(c'(\gamma'_1,\ldots,\gamma'_k)) = \xi_{2g(c'(\gamma'_1,\ldots,\gamma'_k))}$ for $c' \in C'_k \setminus C_k$ and $\gamma'_1,\ldots,\gamma'_k \in L(C')$.

Observe that looking at the index of a variable in $\tau(L(C'))$ we can decide whether it comes from a variable or a formula starting with a connective in $C' \setminus C$.

On the other hand, let $\tau_g^{-1} : \Xi \to L(C')$ be the following substitution:

- $\tau_g^{-1}(\xi_{2i+1}) = \xi_i$ for $\xi_i \in \Xi$;
- $\tau_g^{-1}(\xi_{2i}) = g^{-1}(i)$.

In the sequel, we fix a bijective map g and omit the reference to g and just write τ. The following lemma is proved using a straightforward induction.

Lemma 2.12
If $C \subseteq C'$, then $\tau^{-1}\circ\tau = id$ and $\tau\circ\tau^{-1} = id$.

We are ready to define fibring of consequence systems. We start with some notation. Assume that $C \subseteq C'$ and $\langle C,\vdash\rangle$ is a consequence system. For each $\Gamma' \subseteq L(C')$ we define its closure as follows: $\Gamma'^{\vdash} = \tau^{-1}(\tau(\Gamma')^{\vdash})$, where τ is the translation map from $L(C')$ to $L(C)$. We assume given a bijective map $g : L(C') \cup L(C'') \to \mathbb{N}$ and denote by $\tau'_g : L(C') \cup L(C'') \to L(C')$ and $\tau''_g : L(C') \cup L(C'') \to L(C'')$ the corresponding translation maps. We use τ' and τ'' instead of τ'_g and τ''_g.

The *fibring of consequence systems* $\mathcal{C}' = \langle C',\vdash'\rangle$ and $\mathcal{C}'' = \langle C'',\vdash''\rangle$ is a pair $\mathcal{C}' \uplus \mathcal{C}'' = \langle C,\vdash\rangle$ where

- $C = C' \cup C''$;
- $\vdash : \wp L(C) \to \wp L(C)$ where, for each $\Gamma \subseteq L(C)$, $\Gamma^{\vdash}$ is inductively defined as follows:
  1. $\Gamma \subseteq \Gamma^{\vdash}$;
  2. If $\Delta \subseteq \Gamma^{\vdash}$ then $\Delta^{\vdash'} \cup \Delta^{\vdash''} \subseteq \Gamma^{\vdash}$;

using the translation maps τ' and τ''.

The fibring is said to be *unconstrained* when $C' \cap C'' = \emptyset$; otherwise is *constrained.* Fibring can be seen as a "limit" construction over the class of quasi consequence systems.

Proposition 2.13
Consider the following transfinite sequence of quasi consequence systems:

- $\mathcal{C}_0 = \langle C' \cup C'', \vdash_0 \rangle$ *where* $\Gamma^{\vdash_0} = \Gamma$ *for every* $\Gamma \subseteq L(C)$*;*
- $\mathcal{C}_{\beta+1} = \langle C' \cup C'', \vdash_{\beta+1} \rangle$ *where* $\Gamma^{\vdash_{\beta+1}} = \tau'^{-1}(\tau'(\Gamma^{\vdash_\beta})^{\vdash'}) \cup \tau''^{-1}(\tau''(\Gamma^{\vdash_\beta})^{\vdash''})$ *for every* $\Gamma \subseteq L(C)$*;*
- $\mathcal{C}_\alpha = \langle C' \cup C'', \vdash_\alpha \rangle$ *where* $\Gamma^{\vdash_\alpha} = \bigcup_{\beta<\alpha} \Gamma^{\vdash_\beta}$ *if* α *is a limit ordinal.*

Then $\mathcal{C}' \uplus \mathcal{C}'' = \mathcal{C}_\alpha$ *for some ordinal* α.

Proof. The operator $\Upsilon : \wp L(C) \to \wp L(C)$ such that $\Upsilon(\Delta) = \Delta^{\vdash'} \cup \Delta^{\vdash''}$ is monotonic and extensive, that is, $\Gamma \subseteq \Upsilon(\Gamma)$, over the complete lattice $\langle \wp L(C), \subseteq \rangle$. Hence Υ satisfies Tarski's fixed point theorem and so, for each Γ, there is a least fixed point $\Gamma^{\vdash_\alpha}$. It is easy to see that $\Gamma^{\vdash_\alpha} = \Gamma^{\vdash}$. ∎

A sufficient condition can be given stating when this construction is finite.

Proposition 2.14
Let $\mathcal{C}'$ *and* $\mathcal{C}''$ *be compact consequence systems. Then*

$$\mathcal{C}' \uplus \mathcal{C}'' = \bigcup_{i \in \mathbb{N}} \mathcal{C}_i.$$

Proof. It is enough to show that the operator $\Upsilon : \wp L(C) \to \wp L(C)$ defined in Proposition 2.13 is continuous w.r.t. the same order as before, and so Kleene's fixed point theorem can be applied. The operator Υ is continuous if it preserves directed unions. Let $\{\Delta_a\}_{a\in A}$ be a family of sets in $L(C)$ with A a directed set. Monotonicity implies that

$$\bigcup_{a\in A} \Upsilon(\Delta_a) \subseteq \Upsilon(\bigcup_{a\in A} \Delta_a).$$

It remains to show the other inclusion. Let $\varphi \in \Upsilon(\bigcup_{a\in A} \Delta_a)$. Assume, with no loss of generality, that $\varphi \in (\bigcup_{a\in A} \Delta_a)^{\vdash'}$. Then $\varphi \in \tau'^{-1}(\tau'(\bigcup_{a\in A} \Delta_a)^{\vdash'})$ and so there is $\varphi' \in (\tau'(\bigcup_{a\in A} \Delta_a))^{\vdash'}$ such that $\tau'^{-1}(\varphi')$ is φ. Since $\langle C' \vdash' \rangle$ is compact there is $B \subseteq A$ finite such that $\varphi' \in (\tau'(\bigcup_{b\in B} \Delta_b))^{\vdash'}$. Since A is a directed set, there is $d \in A$ such that $\Delta_b \subseteq \Delta_d$ for every $b \in B$, and so $\varphi' \in \tau'(\Delta_d)$. Therefore $\varphi \in \tau'^{-1}(\tau'(\Delta_d))$. Applying Kleene's fixed point theorem, we conclude that $\Gamma^{\vdash} = \bigcup_{n\in\mathbb{N}} \Upsilon^n(\Gamma)$. ∎

In general, we can still place an upper bound on the cardinality of α.

Proposition 2.15
With the notation of Proposition 2.13, α *is countable.*

Proof. The sequence $\Gamma^{\vdash_0}, \Gamma^{\vdash_1}, \ldots, \Gamma^{\vdash_\alpha}$ is strictly increasing, hence $|\Gamma^{\vdash_\beta}| \geq |\beta|$ for each $\beta = 0, \ldots, \alpha$. Since $\Gamma^{\vdash} \subseteq L(C)$ and $L(C)$ is countable, it follows that α must also be countable. ∎

We now show that the fibring of two consequence systems is a consequence system and moreover that the consequence operator is related with the consequence operators of the original consequence systems.

PROPOSITION 2.16
Fibring $\mathcal{C}' \uplus \mathcal{C}''$ is a consequence system. Moreover

$$\mathcal{C}' \leq \mathcal{C}' \uplus \mathcal{C}'' \text{ and } \mathcal{C}'' \leq \mathcal{C}' \uplus \mathcal{C}''.$$

PROOF. (a) $\mathcal{C}' \uplus \mathcal{C}''$ is a consequence system.

(1) Extensivity: $\Gamma \subseteq \Gamma^{\vdash}$. Let $\gamma \in \Gamma$. Then $\tau'(\gamma) \in \tau'(\Gamma)$. By extensivity of $\vdash'$, $\tau'(\gamma) \in (\tau'(\Gamma))^{\vdash'}$; thus $\tau'^{-1}(\tau'(\gamma)) \in \tau'^{-1}((\tau'(\Gamma))^{\vdash'})$, hence $\gamma \in \tau'^{-1}((\tau'(\Gamma))^{\vdash'})$ using Lemma 2.12. Therefore $\gamma \in \Gamma^{\vdash'}$ and so $\gamma \in \Gamma^{\vdash}$.
(2) Monotonicity. Let $\Gamma_1 \subseteq \Gamma_2$. Then $\tau'(\Gamma_1) \subseteq \tau'(\Gamma_2)$ and $\tau''(\Gamma_1) \subseteq \tau''(\Gamma_2)$, and by monotonicity of $\vdash'$ and $\vdash''$, $(\tau'(\Gamma_1))^{\vdash'} \subseteq (\tau'(\Gamma_2))^{\vdash'}$ and $(\tau''(\Gamma_1))^{\vdash''} \subseteq (\tau''(\Gamma_2))^{\vdash''}$. Hence $\Gamma_1^{\vdash} \subseteq \Gamma_2^{\vdash}$.
(3) Idempotence. By definition of $\vdash$ there is α such that $(\Gamma^{\vdash})^{\vdash} = (\Gamma^{\vdash})^{\vdash_\alpha}$. We show by induction that $(\Gamma^{\vdash})^{\vdash_\alpha} \subseteq \Gamma^{\vdash}$ for every α. (i) $\alpha = 0$. Obvious. (ii) $\alpha = \beta + 1$. By induction hypothesis $(\Gamma^{\vdash})^{\vdash_\beta} \subseteq \Gamma^{\vdash}$ and so, by definition of $\Gamma^{\vdash}$, we have $(((\Gamma^{\vdash})^{\vdash_\beta})^{\vdash'} \cup ((\Gamma^{\vdash})^{\vdash_\beta})^{\vdash''}) \subseteq \Gamma^{\vdash}$ which leads, by definition of $\vdash$, to $(\Gamma^{\vdash})^{\vdash_\alpha} \subseteq \Gamma^{\vdash}$. (iii) α is a limit ordinal. Straightforward.
(4) Closure for renaming substitutions. Let ρ be a renaming substitution. For each $\Gamma \subseteq L(C' \cup C'')$ there is α such that $\Gamma^{\vdash} = \Gamma^{\vdash_\alpha}$. It is enough to prove by induction on α that $\rho(\Gamma^{\vdash_\alpha}) \subseteq (\rho(\Gamma))^{\vdash}$. (i) $\alpha = 0$. Then $\rho(\Gamma) \subseteq (\rho(\Gamma))^{\vdash}$ by extensivity. (ii) $\alpha = \beta + 1$. Then $\rho(\Gamma^{\vdash_\alpha}) = \rho((\Gamma^{\vdash_\beta})^{\vdash'} \cup (\Gamma^{\vdash_\beta})^{\vdash''}) \subseteq (\rho(\Gamma^{\vdash_\beta}))^{\vdash'} \cup (\rho(\Gamma^{\vdash_\beta}))^{\vdash''}$ since $\mathcal{C}'$ and $\mathcal{C}''$ are closed for renaming substitutions. By induction hypothesis, $\rho(\Gamma^{\vdash_\beta}) \subseteq (\rho(\Gamma))^{\vdash}$ and so both $(\rho(\Gamma^{\vdash_\beta}))^{\vdash'} \subseteq (\rho(\Gamma))^{\vdash}$ and $(\rho(\Gamma^{\vdash_\beta}))^{\vdash''} \subseteq (\rho(\Gamma))^{\vdash}$. (iii) α is a limit ordinal. Straightforward.

(b) Since $C' \subseteq C' \cup C''$, it remains to show that $\Gamma'^{\vdash'} \subseteq \Gamma'^{\vdash}$ for $\Gamma' \subseteq L(C')$, and similarly for $\mathcal{C}''$. Assume that $\varphi' \in \Gamma'^{\vdash'}$. Let $\rho : \Xi \to L(C')$ be a renaming substitution such that $\rho(\xi_i) = \xi_{2i+1}$ for every $\xi_i \in \Xi$. Since $\mathcal{C}'$ is closed for renaming substitutions, $\rho(\varphi') \in (\rho(\Gamma'))^{\vdash'}$. Observing that ρ coincides with τ' for formulas in $L(C')$, we have $\tau'(\varphi') \in (\tau'(\Gamma'))^{\vdash'}$, hence $\varphi' \in \tau'^{-1}((\tau'(\Gamma'))^{\vdash'})$ and so $\varphi' \in \Gamma^{\vdash}$. ∎

The following result shows that the closure in $\mathcal{C}' \uplus \mathcal{C}''$ of a set of formulas Γ' in $L(C')$ is the same as the closure in $\mathcal{C}'$ of Γ'. The same applies to $\mathcal{C}''$. As pointed out by Gabbay in [12], this is a key requirement for a good definition of fibring.

PROPOSITION 2.17
Unconstrained fibring $\mathcal{C}' \uplus \mathcal{C}''$ of non-trivial consequence systems closed under substitution is conservative, that is $\Gamma'^{\vdash} = \Gamma'^{\vdash'}$ for every $\Gamma' \subseteq L(C')$ and $\Gamma''^{\vdash} = \Gamma''^{\vdash''}$ for every $\Gamma'' \subseteq L(C'')$.

PROOF. There is α such that $\Gamma'^{\vdash} = \Gamma'^{\vdash_\alpha}$. We show by induction that $\Gamma'^{\vdash_\alpha} \subseteq \Gamma'^{\vdash'}$ for every α. (i) $\alpha = 0$. Then $\Gamma' \subseteq \Gamma'^{\vdash'}$ by extensivity of $\vdash'$. (ii) $\alpha = \beta + 1$. We have two cases. (1) $\varphi' \in \tau'^{-1}((\tau'(\Gamma'^{\vdash_\beta}))^{\vdash'})$. Since $\mathcal{C}'$ is closed under substitution, it follows that $\tau'^{-1}((\tau'(\Gamma'^{\vdash_\beta}))^{\vdash'}) \subseteq (\tau'^{-1}((\tau'(\Gamma'^{\vdash_\beta}))))^{\vdash'}$, hence $\varphi' \in (\tau'^{-1}((\tau'(\Gamma'^{\vdash_\beta}))))^{\vdash'}$ and by Lemma 2.12, $\varphi' \in (\Gamma'^{\vdash_\beta})^{\vdash'}$. On the other hand, by the induction hypothesis, $\Gamma'^{\vdash_\beta} \subseteq$

$\Gamma^{\vdash'}$ and so monotonicity and idempotence of $\vdash'$, $\varphi \in \Gamma'^{\vdash'}$. (2) $\varphi' \in \tau''^{-1}((\tau''(\Gamma'))^{\vdash''})$. Then there is $\varphi'' \in (\tau''(\Gamma'))^{\vdash''}$ such that $\tau''^{-1}(\varphi'')$ is φ'. Observe that, since $C' \cap C'' = \emptyset$, φ'' must be a variable. Since $\mathcal{C}''$ is non-trivial, the only variables in $\tau''^{-1}((\tau''(\Gamma'))^{\vdash''})$ are those already in $\tau''(\Gamma')$, so $\varphi'' \in \tau''(\Gamma')$ and hence $\varphi' \in \Gamma'$. ∎

Fibring plays a special in the class of consequence systems as stated in below. We need before an auxiliary result.

Proposition 2.18
The fibring of consequence systems that are closed for substitution is also closed for substitution.

Proof. The basic step is to show by induction that $\sigma(\Gamma^{\vdash_\alpha}) \subseteq (\sigma(\Gamma))^{\vdash}$. ∎

Proposition 2.19
Fibring is the supremum in the class of consequence systems closed for substitution.

Proof. By Propositions 2.16 and 2.18, one only has to prove that $\mathcal{C}' \uplus \mathcal{C}'' \leq \mathcal{C}'''$ whenever $\mathcal{C}' \leq \mathcal{C}'''$ and $\mathcal{C}'' \leq \mathcal{C}'''$. Let $\mathcal{C}'''$ be such a system; we have to show that $\Gamma^{\vdash} \subseteq \Gamma^{\vdash'''}$. Since there is α such that $\Gamma^{\vdash} = \Gamma^{\vdash_\alpha}$ we show by induction that $\Gamma^{\vdash_\alpha} \subseteq \Gamma^{\vdash'''}$ for every α. (i) $\alpha = 0$. Then $\Gamma \subseteq \Gamma^{\vdash'''}$, by extensivity of $\vdash'''$. (ii) $\alpha = \beta+1$. We have to show that $\tau'^{-1}(\tau'(\Gamma^{\vdash_\beta}))^{\vdash'} \subseteq \Gamma^{\vdash'''}$ and similarly $\tau''^{-1}(\tau''(\Gamma^{\vdash_\beta}))^{\vdash''} \subseteq \Gamma^{\vdash'''}$. By the induction hypothesis $\Gamma^{\vdash_\beta} \subseteq \Gamma^{\vdash'''}$, hence $\tau'(\Gamma^{\vdash_\beta}) \subseteq \tau'(\Gamma^{\vdash'''})$ and so, by monotonicity of $\vdash'$, $(\tau'(\Gamma^{\vdash_\beta}))^{\vdash'} \subseteq (\tau'(\Gamma^{\vdash'''}))^{\vdash'}$. But $\mathcal{C}' \leq \mathcal{C}'''$ by hypothesis hence $(\tau'(\Gamma^{\vdash'''}))^{\vdash'} \subseteq (\tau'(\Gamma^{\vdash'''}))^{\vdash'''}$ and therefore $(\tau'(\Gamma^{\vdash_\beta}))^{\vdash'} \subseteq (\tau'(\Gamma^{\vdash'''}))^{\vdash'''}$. Moreover $\tau'^{-1}((\tau'(\Gamma^{\vdash_\beta}))^{\vdash'}) \subseteq \tau'^{-1}((\tau'(\Gamma^{\vdash'''}))^{\vdash'''})$, then, since $\mathcal{C}'''$ is closed for substitution, $\tau'^{-1}((\tau'(\Gamma^{\vdash_\beta}))^{\vdash'}) \subseteq (\tau'^{-1}(\tau'(\Gamma^{\vdash'''})))^{\vdash'''}$, hence $\tau'^{-1}((\tau'(\Gamma^{\vdash_\beta}))^{\vdash'}) \subseteq (\Gamma^{\vdash'''})^{\vdash'''}$ and so $\tau'^{-1}(\tau'(\Gamma^{\vdash_\beta}))^{\vdash'} \subseteq \Gamma^{\vdash'''}$, by idempotence of $\vdash'''$. (iii) The case where α is a limit ordinal is straightforward. ∎

Now we can give a first attempt to solve the problem of heterogeneous fibring at the deductive level. The basic idea is that once we are given two calculi, of the same kind or not, we extract the induced consequence systems thus getting a homogeneous scenario. Afterwards we obtain the consequence system that represents the fibring of the induced consequence systems.

Example 2.20
Assume that we start with a Hilbert calculus H' and a sequent calculus G''. Their fibring is the consequence system

$$\mathcal{C}(H') \uplus \mathcal{C}(G'').$$

Note that the notion of derivation (finite sequence of either formulas or sequents) does not play a role in the construction. ◁

From a deductive point of view this solution to heterogeneous fibring is not entirely acceptable because the central notion of derivation as a finite sequence is lost. We do not have the notion of derivation in the fibring. Therefore we prefer to introduce, in the next section, the new notion of proof system to solve the problem of fibring heterogeneous proof system.

However, fibring of consequence systems is a good abstraction if we want to combine a semantic system with a deductive calculus as in the following example.

Example 2.21

Let G_{S4} be the sequent calculus for modal logic S4 as presented in Example 2.6 with $C'_1 = \{\neg, \Box'\}$, $C'_2 = \{\Rightarrow\}$ and $\mathcal{C}(G_{S4}) = \langle C', \vdash' \rangle$ be the induced consequence system. Let S_B be interpretation system over C'', where $C''_1 = \{\neg, \Box''\}$, $C''_2 = \{\Rightarrow\}$ as introduced in Example 2.11. Observe that the propositional connectives are the same but we have two different necessitation operators. Let $\mathcal{C}(S_B) = \langle C'', \models \rangle$ be the induced consequence system. The fibring of G_{S4} and S_B is the consequence system

$$\mathcal{C}(G_{S4}) \uplus \mathcal{C}(S_B) = \langle C' \cup C'', \vdash \rangle.$$

For instance $(\Box'(\Box''(\Box''(\Diamond''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))))))) \in \emptyset^{\vdash}$. *Indeed:*

- $(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))) \in \emptyset^{\vdash'}$ *and so* $(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))) \in \emptyset^{\vdash_1}$*;*
- $(\Box''(\Box''(\Diamond''(\xi_i)))) \in \{\xi_i\}^{\models}$ *and so* $(\Box''(\Box''(\Diamond''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))))) \in \emptyset^{\vdash_2}$*;*
- $(\Box'\xi_j) \in \{\xi_j\}^{\vdash'}$ *and so* $(\Box'(\Box''(\Box''(\Diamond''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))))))) \in \emptyset^{\vdash_3}$*;*
- $(\Box'(\Box''(\Box''(\Diamond''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))))))) \in \emptyset^{\vdash}$ *since* $\emptyset^{\vdash_3} \subseteq \emptyset^{\vdash}$*;*

with $\xi_i = \tau''((\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))))$ *and* $\xi_j = \tau'((\Box''(\Box''(\Diamond''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))))))$.

Of course one may ask if fibring the consequence systems induced by two interpretation structures is the best solution for solving the problem of heterogeneous fibring at the semantic level. The answer is that there are better ways, namely following an algebraic semantic approach as in [21].

Finally we discuss some preservation results such as preservation of compactness and semi-decidability.

Theorem 2.22

The fibring of compact consequence systems is also a compact consequence system.

Proof. We have to show that $\Gamma^{\vdash} = \bigcup_{\Phi \in \wp_{\text{fin}}\Gamma} \Phi^{\vdash}$. (1) The inclusion from right to left follows directly by extensivity and monotonicity. It remains to prove the inclusion from left to right. (2) Let $\varphi \in \Gamma^{\vdash}$. We prove by induction on α that if $\varphi \in \Gamma^{\vdash_\alpha}$ then there is $\Phi \subseteq \Gamma$ finite such that $\varphi \in \Phi^{\vdash}$. (i) $\alpha = 0$. Then $\varphi \in \Gamma^{\vdash_0}$, hence $\varphi \in \Gamma$ and so we can take $\Phi = \{\varphi\}$. (ii) $\alpha = \beta + 1$. Then we have two cases. Without loss of generality, let $\varphi \in (\Gamma^{\vdash_\beta})^{\vdash'}$. Since $\vdash'$ is compact there is $\Psi \subseteq \Gamma^{\vdash_\beta}$ finite such that $\varphi \in \Psi^{\vdash'}$ and moreover by definition of $\vdash$ also $\varphi \in \Psi^{\vdash}$. But, by the induction hypothesis, for each $\psi \in \Psi$ there is $\Phi_\psi \subseteq \Gamma$ finite such that $\psi \in \Phi_\psi^{\vdash}$. Take $\Phi = \bigcup_{\psi \in \Psi} \Phi_\psi$. Then Φ is a finite set such that $\Phi \subseteq \Gamma$ and $\varphi \in \Phi^{\vdash}$. (iii) The case where α is a limit ordinal is straightforward. ∎

We cannot prove preservation of semi-decidability in general. However, we can assume a very generic hypothesis satisfied by any "reasonable" consequence system; in other words, counter-examples will be very strange looking systems.

Theorem 2.23

Let $\langle C', \vdash' \rangle$ and $\langle C'', \vdash'' \rangle$ be strongly semi-decidable consequence systems and let $\mathcal{C} = \langle C, \vdash \rangle$ be their fibring. Assume that, for each recursively enumerable $\Gamma \subseteq L(C)$, $\Gamma^{\vdash} = \Gamma^{\vdash_\alpha}$ for some $\alpha < \omega_1^{CT}$, where ω_1^{CT} is the Church–Kleene ordinal [6] (in other words, α is recursively enumerable). Then $\mathcal{C}$ is a strongly semi-decidable consequence system.

Proof. Let $\Gamma \subseteq L(C') \cup L(C'')$ be a recursively enumerable set. By hypothesis there is a recursively enumerable α such that $\Gamma^{\vdash} = \Gamma^{\vdash_\alpha}$; we show that $\Gamma^{\vdash_\beta}$ is recursively enumerable for every $\beta \leq \alpha$ by induction. (i) $\alpha = 0$. Then $\Gamma_0 = \Gamma$, which is by hypothesis a recursively enumerable set. (ii) Assume $\Gamma^{\vdash_\beta}$ is a recursively enumerable set. Since τ' is recursive, $\tau'(\Gamma^{\vdash_\beta})$ is also recursively enumerable, and so $\tau'(\Gamma^{\vdash_\beta})^{\vdash'}$ is recursively enumerable, since $\langle C', \vdash' \rangle$ is strongly semi-decidable. Finally τ'^{-1} is again recursive, so $\tau'^{-1}(\tau'(\Gamma^{\vdash_\beta})^{\vdash'})$ is recursively enumerable. The same reasoning shows that $\tau''^{-1}(\tau''(\Gamma^{\vdash_\beta})^{\vdash''})$ is recursively enumerable, and since the union of two recursively enumerable sets is recursively enumerable we conclude that $\Gamma^{\vdash_{\beta+1}}$ is recursively enumerable. Finally, let $\epsilon \leq \alpha$ be a limit ordinal. Then $\Gamma^{\vdash_\epsilon} = \bigcup_{\beta<\epsilon} \Gamma^{\vdash_\beta}$. Since ϵ is recursively enumerable, there is a recursive enumeration of $\{\beta : \beta < \epsilon\}$, and hence $\Gamma^{\vdash_\epsilon}$ is a recursive union of recursively enumerable sets and therefore is a recursively enumerable set. ∎

Corollary 2.24
The fibring of semi-decidable compact consequence systems is a semi-decidable consequence system.

Proof. By Proposition 2.2, if $\mathcal{C}'$ and $\mathcal{C}''$ are semi-decidable and compact, then they are both strongly semi-decidable. By Theorem 2.23 (which we can apply since in this case $\alpha = \omega$ by Proposition 2.14), their fibring $\mathcal{C} = \mathcal{C}' \uplus \mathcal{C}''$ is also strongly semi-decidable, hence in particular semi-decidable. ∎

3 Abstract proof systems

The section is dedicated to (abstract) proof systems, which abstract from the usual syntactic presentations of logics keeping the notion of derivation or certificate.

The notion we present is intentionally very abstract. A proof system (defined below) is simply a set of formulas together with a set of derivations about which very little is assumed. Thus, derivations can be sequences of formulas with some internal structure (as is the case, for example, in Hilbert calculi) or bear little or no relationship with the language. We show two examples of the latter, one where derivations are natural numbers and another where there is just one derivation.

A consequence of having such a general definition is that we cannot state in general properties of the (eventual) structure of the derivations and analyze their preservation through fibring. We do not feel this to be a problem at this stage, however, since we are dealing with a more basic question – namely, how to construct a heterogeneous fibring of two logics. Other properties (for example, size of the derivations) can also be studied by looking at subclasses of proof systems generated by specific mechanisms; we intend to do this in future work on this topic.

3.1 Basics

Given a binary relation $S \subseteq A \times B$, we will use the notation $S(a, b)$ to indicate that $(a, b) \in S$ or to say that $S(a, b) = 1$ when viewing the relation as a map $1_S : A \times B \to \{0, 1\}$.

A *proof system* is a tuple $\mathcal{P} = \langle C, D, \circ, P \rangle$ where C is a signature, D is a set, $\circ : \wp(D) \times D \to D$ is a map and $P = \{P_\Gamma\}_{\Gamma \subseteq L(C)}$ is a family of relations $P_\Gamma \subseteq D \times L(C)$

satisfying the following properties, where $P_\Gamma(E,\Psi)$ holds if for every $\psi \in \Psi$ there is $e \in E$ such that $P_\Gamma(e,\psi)$ holds:

- Right reflexivity: $P_\Gamma(D,\Gamma)$ for every $\Gamma \subseteq L(C)$;
- Monotonicity: $P_{\Gamma_1} \subseteq P_{\Gamma_2}$ for every $\Gamma_1 \subseteq \Gamma_2 \subseteq L(C)$;
- Compositionality: Let $\Gamma \cup \{\varphi\} \subseteq L(C)$:
  - $\emptyset \circ d = d$ for every $d \in D$;
  - If $E \subseteq D$ is a non-empty set and there is $\Psi \in \wp L(C)$ such that $P_\Gamma(E,\Psi)$ and $P_\Psi(d,\varphi)$ hold then $P_\Gamma(E \circ d,\varphi)$ also holds;
- Variable exchange: $P_\Gamma(D,\varphi) = P_{\rho(\Gamma)}(D,\rho(\varphi))$ for any (renaming) substitution ρ, that is, a substitution such that $\rho(\xi) \in \Xi$ for every $\xi \in \Xi$.

The set D can be seen as the set of possible *derivations*, $\circ$ is a constructor that returns a derivation given a set of derivations and a derivation and, $P_\Gamma(d,\psi)$ holds when d is a derivation of ψ from the set of formulas Γ. A tuple $\mathcal{P} = \langle C, D, \circ, P\rangle$ is a *quasi-proof system* if all the properties of a proof system hold with the possible exception of compositionality.

A particular (though not so interesting) proof system is the one where $D = L(C)$ and $P_\emptyset(\varphi,\varphi)$ for every $\varphi \in L(C)$. Another example is when we consider $D = L(C)^*$ (that is, D is the set of all finite sequences of formulas), $P_\Gamma(w,\gamma)$ if $\gamma \in \Gamma$ and is the last element of w. We stress that D does not need to be related to C and to the formulas in $L(C)$; for instance, D can be the set of natural numbers. Other examples will be discussed below.

Proposition 3.1
The following properties hold in a proof system:

- *Falsehood:* $P_\Gamma(\emptyset,\varphi) = 0$ *for every* $\varphi \in L(C)$*;*
- *Monotonicity on the first argument:* $P_\Gamma(E_1,\Psi) \leq P_\Gamma(E_2,\Psi)$ *for all* $E_1 \subseteq E_2 \subseteq D$ *and* $\Gamma, \Psi \subseteq L(C)$*;*
- *Anti-monotonicity on the second argument:* $P_\Gamma(E,\Psi_2) \leq P_\Gamma(E,\Psi_1)$ *for all* $E \subseteq D$ *and* $\Psi_1 \subseteq \Psi_2 \subseteq L(C)$*;*
- *Union:* $P_\Gamma(E,\Psi_1 \cup \Psi_2) = P_\Gamma(E,\Psi_1) \times P_\Gamma(E,\Psi_2)$.

Proof. All the properties follow directly from the definitions. ∎

Proof systems sometimes have more properties. A proof system is said to be *non-trivial* if $P_\Pi(D,\xi) = 0$ for any $\Pi \subseteq \Xi$ and $\xi \in \Xi \setminus \Pi$. A proof system is *closed for substitution* if $P_\Gamma(D,\varphi) \leq P_{\sigma(\Gamma)}(D,\sigma(\varphi))$ for each substitution σ. A proof system is said to be *compact or finitary* if for every Γ and ψ there is $\Phi \subseteq \Gamma$ finite such that $P_\Gamma(D,\psi) \leq P_\Phi(D,\psi)$. A proof system is said to be *effective* or *decidable* if P_Γ is recursive for each recursive set $\Gamma \subseteq L(C)$. It is said to be *strongly effective* or *strongly decidable* if P_Γ is recursively enumerable for each recursively enumerable set $\Gamma \subseteq L(C)$.

At this stage, we will not consider the question of whether the set of theorems of a given proof system is recursive. Although this is an interesting aspect, our motivation is applications in automated reasoning, where the most important question is whether a proof can be *checked*. This is the concept we capture with the notion of effective proof system.

Proof systems can be related. We say that $\mathcal{P} = \langle C, D, \circ, P\rangle$ is *weaker* than $\mathcal{P}' = \langle C', D', \circ', P'\rangle$, indicated by $\mathcal{P} \leq \mathcal{P}'$, if $C \subseteq C'$ and $P_\Gamma(D, \varphi) \leq P'_\Gamma(D', \varphi)$ for every $\Gamma \cup \{\varphi\} \subseteq L(C)$. Hence a proof system is weaker than another when it proves less formulas from the same set of formulas.

3.2 Induced proof systems

As in the previous section, we now show that many of the usual syntactic presentations of deductive systems can be seen to induce proof systems. The semantic presentations of logics cannot in general be presented as proof systems because they lack the notion of derivation.

3.2.1 Hilbert calculus

Recall that a Hilbert calculus is a pair $\langle C, R\rangle$ such that C is a signature and R is a set of finitary rules (pairs whose first component is a finite set of formulas and whose second component is a formula). We need some auxiliary notation. Let $\pi(e)$ denote the last element of sequence $e \in L(C)^*$, by $\pi(E)$ the set $\{\pi(e) : e \in E\}$ where $E \subseteq L(C)^*$ and by $d_E^{\pi(E)}$ the sequence obtained by replacing in $d \in L(C)^*$ the last element of sequence e by sequence e, for every $e \in E$[2].

As an illustration assume that d is $\varphi_1 \dots \varphi_i \psi_k \varphi_{i+1} \dots \varphi_n$ and $E = \{\psi_1 \dots \psi_k\}$. Then $d_E^{\pi(E)}$ is the sequence $\varphi_1 \dots \varphi_i \psi_1 \dots \psi_k \varphi_{i+1} \dots \varphi_n$.

Proposition 3.2
A Hilbert calculus $H = \langle C, R\rangle$ induces a compact proof system $\mathcal{P}(H) = \langle C, D, \circ, P\rangle$ as follows:

- $D = L(C)^*$;
- $E \circ d = d_E^{\pi(E)}$;
- $P_\Gamma(d, \psi)$ *holds iff* d *is a Hilbert-derivation for* ψ *from* Γ.

Proof. (a) Right reflexivity. It follows from the fact that the sequence γ is a derivation of γ from set Γ whenever $\gamma \in \Gamma$. (b) Monotonicity. Assume that d is a derivation of φ from Γ_1 and that $\Gamma_1 \subseteq \Gamma_2$. Then d is also a derivation of φ from Γ_2. (c) Compositionality. Assume that $P_\Gamma(E, \Psi)$ and $P_\Psi(d, \varphi)$ hold for some set Ψ. Then $d^\Psi_{E_\Psi}$, where $E_\Psi \subseteq E$ is the set of derivations in E of each $\psi \in \Psi$ from Γ, is a derivation of φ from Γ. Observe that only a finite number of elements of Ψ are used, so $d^\Psi_{E_\Psi}$ is still a finite sequence. (d) Variable exchange. Assume that d is a derivation of φ from Γ and that ρ is a renaming substitution. Then $\rho(d)$ is a derivation of $\rho(\varphi)$ from $\rho(\Gamma)$. ■

It is easy to see that $\mathcal{P}(H)$ is also closed for substitution translating each derivation with the given substitution.

We give a necessary and sufficient condition for $\mathcal{P}(H)$ to be effective.

Theorem 3.3
The proof system $\mathcal{P}(H)$ induced by a Hilbert system $H = \langle C, R\rangle$ such that $\langle L(C), g\rangle$ is a Gödel domain is effective iff the following relations are recursive:

[2] This is, strictly speaking, not well-defined, since there may be several e with the same last element. This is not a problem as long as one assumes that e is chosen uniformly, e.g. choosing the first derivation in a lexicographical ordering.

- *$ax \subseteq L(C)$, such that $ax(\alpha)$ iff α is an instance of an axiom;*
- *$ir_{k+1} \subseteq L(C)^{k+1}$ such that $ir_{k+1}(\alpha_1, \ldots, \alpha_k, \beta)$ iff $\alpha_1, \ldots, \alpha_k, \beta$ is an instance of a $k+1$-ary inference rule.*

PROOF. Let $\mathrm{hyp}_\Gamma \subseteq L(C)$ be such that $\mathrm{hyp}_\Gamma(\gamma)$ iff $\gamma \in \Gamma$.
(1) Assume that P_Γ is recursive whenever Γ is a recursive set. In particular, P_Φ is recursive for every finite set Φ.
(a) The relation ax is recursive since $\mathrm{ax}(\alpha) = P_\emptyset(\langle\alpha\rangle, \alpha)$.
(b) Let $\langle\Theta, \eta\rangle$ be a k-ary rule. The relation ir_{k+1} is recursive since

$$ir_{k+1}(\alpha_1, \ldots, \alpha_k, \beta) = P_\Theta(\langle\alpha_1, \ldots, \alpha_k, \beta\rangle, \beta)$$

where $\Theta = \{\alpha_1, \ldots, \alpha_k\}$.
(2) Let $\Gamma \subseteq L(C)$ be a recursive set. The relation P_Γ such that:

$$P_\Gamma(\omega, \beta) = \qquad (((\omega = \langle\beta\rangle) \wedge (\mathrm{ax}(\beta) \vee \mathrm{hyp}_\Gamma(\beta))) \\ \vee \\ \bigvee_{k=1}^{|\omega|} (\exists_{\omega' \sqsubseteq \omega}(\exists_{i_1 \leq |\omega'|}(\ldots(\exists_{i_k \leq |\omega'|} \\ (P_\Gamma(\omega', \mathrm{last}(\omega')) \wedge (\omega = \omega' \cdot \beta) \wedge \mathrm{ir}_{k+1}((\omega')_{i_1}, \ldots, (\omega')_{i_k}, \beta))))))),$$

where $|\omega|$ denotes the length of ω, $\omega' \sqsubseteq \omega$ stands for "ω' is a prefix of ω" and $\cdot$ denotes concatenation, is recursive since the relations ax, ir, hyp, length and last are recursive, being a prefix is decidable, and the conjunction, disjunction and bounded quantification of recursive relations is a recursive relation. ∎

This is not the only proof system that can be generated from a Hilbert calculus. By construction any initial segment of a Hilbert derivation is itself a valid derivation, which motivates the following definition.

EXAMPLE 3.4
Let $H = \langle C, R\rangle$ be a Hilbert calculus. Then $\mathcal{P}'(H)$ is defined as above, except that $P'_\Gamma(d, \varphi)$ now holds iff d is a valid derivation from Γ and φ occurs in d.

The same example can be used to induce a proof system where the set of derivations bears no (apparent) relationship to the language.

EXAMPLE 3.5
Let $H = \langle C, R\rangle$ be a Hilbert calculus and let g be a Gödelization of $L(C)$, that is, $g : \mathbb{N} \to L(C)$ is recursive. Define $\mathcal{P}''(H)$ as follows:

- *D is $\{g(w) : w \in L(C)^*\}$;*
- *$P_\Gamma(n, \varphi)$ if n is the Gödel number of a sequence $w \in L(C)^*$ and w is a Hilbert-derivation of φ from Γ.* ◁

3.2.2 Sequent calculus

Recall that a sequent calculus is a pair $\langle C, R\rangle$ where C is a signature and R is a set of rules (pairs whose first component is a finite set of sequents and whose second component is a sequent): structural rules and, for each connective, a right and a left rule. We need some auxiliary notation. Assume that d is a sequence of sequents with initial sequent $\Delta_1 \to \Delta_2$. When the initial sequent is important we can use $d_{\Delta_1\to\Delta_2}$ to refer to d.

PROPOSITION 3.6
A sequent calculus $G = \langle C, R\rangle$ induces a proof system $\mathcal{P}(G) = \langle C, D, \circ, P\rangle$ defined as follows:

- $D = \mathrm{Seq}(C)^*$ *where* $\mathrm{Seq}(C)$ *is the set of all sequents defined with formulas in* $L(C)$*;*
- *Let* $E \cup \{d_{\Theta\to\varphi}\} \subseteq D$ *where* $E \neq \emptyset$*. Let* $\{\theta_1, \dots, \theta_n\} \subseteq \Theta$ *be the set of all sequents such that* $d_{\Gamma_i\to\theta_i} \in E$ *for every* $i = 1, \dots, n$*. Let* $\bar{\Theta} = \Theta \setminus \{\theta_1, \dots, \theta_n\}$*. Then* $E \circ d_{\Theta\to\varphi}$ *is the following sequence*[3]*:*

	$\bar{\Theta}, \Gamma_1, \dots, \Gamma_n \to \varphi$	Cut *1a,1b*
1a	$\bar{\Theta}, \Gamma_1, \dots, \Gamma_n \to \varphi, \theta_1$	LW*, RW
	$d_{\Gamma_1\to\theta_1}$	
1b	$\theta_1, \bar{\Theta}, \Gamma_1, \dots, \Gamma_n \to \varphi$	Cut *2a,2b*
	$\vdots$	
	$\theta_1, \dots, \theta_{n-1}, \bar{\Theta}, \Gamma_1, \dots, \Gamma_n \to \varphi$	Cut *na,nb*
na	$\theta_1, \dots, \theta_{n-1}, \bar{\Theta}, \Gamma_1, \dots, \Gamma_n \to \varphi, \theta_n$	LW*, RW
	$d_{\Gamma_n\to\theta_n}$	
nb	$\Theta, \Gamma_1, \dots, \Gamma_n \to \varphi$	LW*
	$d_{\Theta\to\varphi}$	

  where LW*, RW *indicate several applications of left weakening followed by right weakening;*
- $\emptyset \circ d = d$*;*
- $P_\Gamma(d, \varphi)$ *holds if* d *is a sequent-derivation of* φ *from* Γ*.*

PROOF. (a) Right reflexivity. Just consider the derivation $\Gamma \to \gamma$ justified as an axiom whenever $\gamma \in \Gamma$. (b) Monotonicity. Consider a derivation $d_{\Gamma_1\to\varphi}$ and $\Gamma_1 \subseteq \Gamma_2$. Then the following is a derivation for $\Gamma_2 \to \varphi$:

1	$\Gamma_2 \to \varphi$	LW* 2
2	$d_{\Gamma_1\to\varphi}$	

(c) Compositionality. Direct from the definition of $\circ$. (d) Variable exchange. If d is a derivation for $\Gamma \to \varphi$ and ρ is a renaming substitution then $\rho(d)$ is a derivation for $\rho(\Gamma) \to \rho(\varphi)$. ∎

REMARK 3.7
Observe that in the case of sequents we can define binary relations $\bar{P_H} \subseteq D \times \mathrm{Seq}(C)$ where H is a set of sequents over $L(C)$ (hence in $\mathrm{Seq}(C)$) but stating that $P_H(d, s) = 1$ whenever $H \vdash_G s$ with sequent-derivation d. Of course $P_\Gamma(d, \varphi)$ is $\bar{P}_\emptyset(d, \Gamma \to \varphi)$.

---

[3] See the footnote on page 17

3.2.3 Tableau calculus

Recall that a tableau calculus is a pair $\langle C, R\rangle$ where C is a signature and R is a set of rules (pairs whose first component is a set of finite sets of labelled formulas and whose second element is a labelled formula): excluded middle and for each connective a positive and a negative rule. We need some notation: if $d_1 \dots d_n$ is a finite sequence of sets and Ψ is a set of labelled formulas, then $\Psi d_1 \dots d_n$ is the finite sequence $d_1 \cup \Psi \dots d_n \cup \Psi$. Observe that if $d_1 \dots d_n$ is a tableau-derivation for $\Gamma \vdash_S \varphi$ then $\Psi d_1 \dots d_n$ is a tableau-derivation for $\Gamma \cup \Psi \vdash_S \varphi$. We use $1:A$ to refer to the set $\{(1\!:\!a) : a \in A\}$.

Proposition 3.8
A tableau calculus $G = \langle C, R\rangle$ induces a proof system $\mathcal{P}(G) = \langle C, D, \circ, P\rangle$ defined as follows:

- $D = (\wp L^{\lambda}(C))^*$ *is the set of all finite sequences of sets of labelled formulas;*
- *Let $E \cup \{d\} \subseteq D$ be such that the first set in d is $1:\Theta \cup \{0:\varphi\}$ and, for $\theta_i \in \Theta$, with $i = 1,\dots,n$, there are $e_i \in E$ whose first set is $1:\Gamma_i \cup \{0:\theta_i\}$. Let $\bar{\Theta} = \Theta \setminus \{\theta_1,\dots,\theta_n\}$ and $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$. Define $E \circ d$ as follows, where Ψ_1 is $1\!:\!\bar{\Theta} \cup (1\!:\!\Gamma \setminus 1\!:\!\Gamma_1) \cup \{0\!:\!\varphi\}$ and Ψ_n is $1\!:\!\bar{\Theta} \cup (1\!:\!\Gamma \setminus 1\!:\!\Gamma_n) \cup \{1\!:\!\theta_1,\dots,1\!:\!\theta_{n-1}, 0\!:\!\varphi\}$:*

	$1\!:\!\bar{\Theta}, 1\!:\!\Gamma, 0\!:\!\varphi$	*EM 1a,1b*
1a	$0\!:\!\theta_1, 1\!:\!\bar{\Theta}, 1\!:\!\Gamma, 0\!:\!\varphi$	
	$\Psi_1 e_1$	
1b	$1\!:\!\theta_1, 1\!:\!\Gamma, 0\!:\!\varphi$	*EM 2a,2b*
	$\vdots$	
	$1\!:\!\theta_1,\dots,1\!:\!\theta_{n-1}, 1\!:\!\Gamma, 0\!:\!\varphi$	*EM na,nb*
na	$1\!:\!\theta_1,\dots,0\!:\!\theta_n, 1\!:\!\Gamma, 0\!:\!\varphi$	
	Ψ_n	
nb	$1\!:\!\Theta, 0\!:\!\varphi$	
	$1\!:\!\Gamma d$	

- $P_\Gamma(d,\varphi)$ *holds iff d is a tableau derivation for φ from Γ.*

Proof. (a) Right reflexivity. The set $1:\Gamma \cup \{0:\gamma\}$ is an absurd when $\gamma \in \Gamma$. (b) Monotonicity. Let $\Gamma_1 \subseteq \Gamma_2$ and d be a tableau-derivation of φ from Γ_1. Then $(\Gamma_2 \setminus \Gamma_1)d$ is a tableau-derivation of φ from Γ_2. (c) Compositionality. Follows from the fact that $E \circ d$ is a tableau-derivation of φ from $\bar{\Theta} \cap \Gamma$ whenever d is a tableau-derivation from Θ and there is a tableau-derivation in E of θ_i from Γ_i for every $i = 1,\dots,n$. (d) Variable exchange. If d is a tableau-derivation of φ from Γ then $\rho(d)$ is a tableau-derivation of $\rho(\varphi)$ from $\rho(\Gamma)$ for every renaming substitution ρ. ∎

Remark 3.9
Observe that in the case of tableau calculi we can define binary relations $\bar{P_H} \subseteq D \times \wp L^{\lambda}(C)$ where H is a set of sets of labelled formulas over $L(C)$ (hence in $\wp L^{\lambda}(C)$) but stating that $P_H(d,s) = 1$ whenever $H \vdash_S s$ with tableau-derivation d. Of course $P_\Gamma(d,\varphi)$ is $\bar{P}_\emptyset(d, 1\!:\!\Gamma \cup \{0\!:\!\varphi\})$.

3.3 Fibring

The *fibring* of two proof systems $\mathcal{P}' = \langle C', D', \circ', P' \rangle$ and $\mathcal{P}'' = \langle C'', D'', \circ'', P'' \rangle$ is the tuple $\mathcal{P}' \uplus \mathcal{P}'' = \langle C, D, \circ, P \rangle$ defined as follows:

- $C = C' \cup C''$;
- D is a set inductively defined as follows:
  - $D' \cup D'' \subseteq D$;
  - If $E \subseteq D$ then $\wp E \times (D' \cup D'') \subseteq D$;
- $E \circ d = \langle E, d \rangle$ if $E \neq \emptyset$ and d otherwise;
- $P_\Gamma(d', \varphi)$ holds if $P'_{\tau'(\Gamma)}(d', \tau'(\varphi))$ holds for $d' \in D'$;
- $P_\Gamma(d'', \varphi)$ holds if $P''_{\tau''(\Gamma)}(d'', \tau''(\varphi))$ holds for $d'' \in D''$;
- $P_\Gamma(\langle E, d \rangle, \varphi)$ holds iff there is a set $\Psi \in L(C)$ for which both $P_\Psi(d, \varphi)$ and $P_\Gamma(E, \Psi)$ hold.

Notice that the second- and third-to-last cases are not mutually exclusive since D' and D'' need not be disjoint.

Before showing that the fibring of proof systems is a proof system we give the intuition behind the construction of the set of derivations and some examples. Take as an example $\langle \{d'\}, d'' \rangle$; this is a derivation provided that d' is a derivation in D' and d'' is a derivation in D''. Such a derivation is only relevant when we use the relation P. Saying that

$$P_\emptyset(\langle \{d'\}, d'' \rangle, c''(c'(\xi_1)))$$

holds means:

- that d'' is a derivation of $c''(\xi_k)$ where $\xi_k = \tau''(c'(\xi_1))$, assuming that we take the singleton $\{\xi_k\}$ as the set of hypotheses, in other words provided that $P''_{\{\xi_k\}}(d'', c''(\xi_k))$ holds;
- and that d' is a derivation of $c'(\xi_1)$ taking the empty set as the set of hypotheses, in other words provided that $P'_\emptyset(d', c'(\xi_1))$ holds.

We now give some examples of fibring involving logics presented with different calculi.

Example 3.10
Consider the proof system $\mathcal{P}(G_{S4}) = \langle C', R' \rangle$ induced by the sequent calculus for modal logic S4 as presented in Example 2.6 and the proof system $\mathcal{P}(H_B) = \langle C'', R'' \rangle$ induced by the Hilbert calculus for modal logic with axiom B as in Example 2.4. They share the propositional connectives, but in the fibring we have two necessitations: an $S4$ $\Box'$ (and consequently an $S4$ diamond $\Diamond'$) and a B $\Box''$ (and consequently a B diamond $\Diamond''$). We can prove in $\mathcal{P}(G_{S4}) \uplus \mathcal{P}(H_B)$ that

$$P_\emptyset(\langle \{\langle \{d'_1\}, d'' \rangle\}, d'_2 \rangle, (\Box'(\Box''(\Diamond''(\Box''(\Diamond'(\xi_1 \Rightarrow (\Box' \xi_1))))))))$$

holds. Indeed

- $P'_{\{\xi_i\}}(d'_2, (\Box'\xi_i))$ *holds in $\mathcal{P}(G_{S4})$ with derivation d'_2 as follows:*

1	$\to (\Box'\xi_i)$	$R\Box'$ *2*
2	$\to \xi_i$	*Hyp*

where ξ_i *is* $\tau'(\Box''(\Diamond''(\Box''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))))))$*;*

- *and we have to show that* $P_\emptyset(\langle\{d_1'\}, d''\rangle, \Box''(\Diamond''(\Box''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1))))))$ *holds;*

But

$$P_\emptyset(\langle\{d_1'\}, d''\rangle, (\Box''(\Diamond''(\Box''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))))))$$

holds since

- $P''_{\{\xi_j\}}(d'', (\Box''(\Diamond''(\Box''(\xi_j)))))$ *holds in* $\mathcal{P}(H_B)$ *with derivation* d'' *as follows:*

1	ξ_j	*Hyp*
2	$(\Box''\xi_j)$	*Nec 1*
3	$((\Box''\xi_j) \Rightarrow (\Box''(\Diamond''(\Box''\xi_j))))$	*B*
4	$(\Box''(\Diamond''(\Box''\xi_j)))$	*MP 2,3*

where ξ_j *is* $\tau''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$*;*

- *and* $P_\emptyset(d_1', \Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$ *holds in* $\mathcal{P}(G_{S4})$ *with derivation* d_1' *as follows:*

1	$\to (\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$	$R\Diamond'$ *2*
2	$\to (\xi_1 \Rightarrow (\Box'\xi_1)), (\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$	$R\!\Rightarrow$ *3*
3	$\xi_1 \to (\Box'\xi_1), (\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$	$R\Box'$ *4*
4	$\to \xi_1, (\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$	$R\Diamond'$ *5*
5	$\to \xi_1, (\xi_1 \Rightarrow (\Box'\xi_1)), (\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$	$R\!\Rightarrow$ *6*
6	$\xi_1 \to \xi_1, (\Box'\xi_1), (\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))$	*Ax*

Hence d_2', d'', d_1' *provide a derivation for* $(\Box'(\Box''(\Diamond''(\Box''(\Diamond'(\xi_1 \Rightarrow (\Box'\xi_1)))))))$ *without any hypotheses. Observe that the number of pairings in the derivation indicates the way we have to use the component proof systems. In the example above we have three pairings and we had to use the component proof systems three times.* ◁

EXAMPLE 3.11

Consider the propositional part of the proof system $\mathcal{P}(G_{S4}) = \langle C', R'\rangle$ *in Example 2.6 (with negation and implication as connectives) and the proof system* $\mathcal{C}(S_{P_{\wedge,\Rightarrow}}) = \langle C'', R''\rangle$ *presented in Example 2.8. We prove in* $\mathcal{P}(G_{S4}) \uplus \mathcal{P}(S_{P_{\wedge,\Rightarrow}})$ *that*

$$P_\emptyset(\langle\{d_1', d_2'\}, d''\rangle, ((\neg(\xi_1 \Rightarrow \xi_2)) \Rightarrow (\xi_1 \wedge (\neg\,\xi_2))))$$

holds. Indeed, taking $\xi_i = \tau''(\neg(\xi_1 \Rightarrow \xi_2))$ *and* $\xi_j = \tau''(\neg\,\xi_2)$*, we have:*

- $P''_{\{(\xi_i \Rightarrow \xi_1),(\xi_i \Rightarrow \xi_j)\}}(d'', (\xi_i \Rightarrow (\xi_1 \wedge \xi_j)))$ *holds in* $S_{P_{\wedge,\Rightarrow}}$ *with derivation* d'' *as follows:*

1.	$1{:}(\xi_i \Rightarrow \xi_1), 1{:}(\xi_i \Rightarrow \xi_j), 0{:}(\xi_i \Rightarrow (\xi_1 \wedge \xi_j))$	$0\!\Rightarrow$ *2*
2.	$1{:}(\xi_i \Rightarrow \xi_1), 1{:}(\xi_i \Rightarrow \xi_j), 1{:}\xi_i, 0{:}(\xi_1 \wedge \xi_j)$	$1\!\Rightarrow$ *3,4*
3.	$0{:}\xi_i, 1{:}(\xi_i \Rightarrow \xi_j), 1{:}\xi_i, 0{:}(\xi_1 \wedge \xi_j)$	*Ax*
4.	$1{:}\xi_1, 1{:}(\xi_i \Rightarrow \xi_j), 1{:}\xi_i, 0{:}(\xi_1 \wedge \xi_j)$	$1\!\Rightarrow$ *5,6*
5.	$1{:}\xi_1, 0{:}\xi_i, 1{:}\xi_i, 0{:}(\xi_1 \wedge \xi_j)$	*Ax*
6.	$1{:}\xi_1, 1{:}\xi_j, 1{:}\xi_i, 0{:}(\xi_1 \wedge \xi_j)$	$0\wedge$ *7,8*
7.	$1{:}\xi_1, 1{:}\xi_j, 1{:}\xi_i, 0{:}\xi_1$	*Ax*
8.	$1{:}\xi_1, 1{:}\xi_j, 1{:}\xi_i, 0{:}\xi_j$	*Ax*

- $P_\emptyset(d_1', ((\neg(\xi_1 \Rightarrow \xi_2)) \Rightarrow \xi_1))$ *holds in* G_{S4} *with derivation* d_1' *as follows:*

1.	$\rightarrow ((\neg(\xi_1 \Rightarrow \xi_2)) \Rightarrow \xi_1)$	$R\!\Rightarrow$ *2*
2.	$(\neg(\xi_1 \Rightarrow \xi_2)) \rightarrow \xi_1$	$L\neg$ *3*
3.	$\rightarrow (\xi_1 \Rightarrow \xi_2), \xi_1$	$R\!\Rightarrow$ *4*
4.	$\xi_1 \rightarrow \xi_2, \xi_1$	Ax

- $P_\emptyset(d'_2, (\neg(\xi_1 \Rightarrow \xi_2)) \Rightarrow (\neg\,\xi_2))$ *holds in* G_{S4} *with derivation* d'_2 *as follows:*

1.	$\rightarrow (((\neg(\xi_1 \Rightarrow \xi_2)) \Rightarrow (\neg\,\xi_2))$	$R\!\Rightarrow$ *2*
2.	$(\neg(\xi_1 \Rightarrow \xi_2)) \rightarrow (\neg\,\xi_2)$	$L\neg$ *3*
3.	$\rightarrow (\xi_1 \Rightarrow \xi_2), (\neg\,\xi_2)$	$R\!\Rightarrow$ *4*
4.	$\xi_1 \rightarrow \xi_2, (\neg\,\xi_2)$	$R\neg$ *5*
5.	$\xi_1, \xi_2 \rightarrow \xi_2$	Ax

Hence d'', d'_1, d'_2 *constitutes a derivation of* $(\neg(\xi_1 \Rightarrow \xi_2)) \Rightarrow (\xi_1 \wedge (\neg\,\xi_2)))$ *with no hypotheses. We have two pairings but in one of them we have to produce two derivations because the corresponding set has two elements.* $\triangleleft$

The following result shows that the set D in the fibring comes out, as expected, as a fixed point construction.

Proposition 3.12
Consider the transfinite sequence of quasi proof systems where D *is replaced by* D_α*.*

- $D_0 = D' \cup D''$*;*
- $D_{\beta+1} = D_\beta \cup (\wp(D_\beta) \times (D' \cup D''))$*;*
- $D_\alpha = \bigcup_{\beta \leq \alpha} D_\beta$ *where* α *is a limit ordinal.*

Then $D = D_\alpha$ *for some ordinal* α*.*

Proof. The operator $\Upsilon : L(C) \rightarrow L(C)$ such that $\Upsilon(\Delta) = \Delta \cup (\wp\Delta \times (D' \cup D''))$ is monotonic over the complete lattice $\langle \wp L(C), \subseteq \rangle$ and $(D' \cup D'') \subseteq \Upsilon(D' \cup D'')$. Hence Υ satisfies Tarski's fixed point theorem and so it has a least fixed point. It is easy to see that this fixed point is D. ∎

The question that arises at this point is when can we obtain each element of D with a finite number of iterations. We give a sufficient condition for that. For this purpose we need to work with a special kind of fibring. The *finite derivation fibring* of proof systems $\mathcal{P}'$ and $\mathcal{P}''$ is the fibring proof system $\mathcal{P}' \uplus \mathcal{P}''$ where D is a set inductively defined as follows:

- $D' \cup D'' \subseteq D$;
- If $E \subseteq D$ then $(\wp_{\mathrm{fin}} E) \times (D' \cup D'') \subseteq D$;

Proposition 3.13
Let $\mathcal{P}' \uplus \mathcal{P}''$ *be the finite derivation fibring of* $\mathcal{P}'$ *and* $\mathcal{P}''$*. Then*

$$D = \bigcup_{i \in \mathbb{N}} D_i.$$

Proof. The operator $\Upsilon' : L(C) \rightarrow L(C)$ such that $\Upsilon'(\Delta) = \Delta \cup (\wp_{\mathrm{fin}}\Delta \times (D' \cup D''))$, analogous to Υ defined in Proposition 3.12, is continuous and so Kleene's fixed point theorem can be applied. ∎

We now provide an example of fibring of two proof systems where we need to compute a transfinite fixed point for obtaining the set of derivations and that is supported by our definition of proof system. This example is admittedly artificial, but it shows that, whenever infinitary rules are allowed, D_ω may not suffice.

EXAMPLE 3.14
Consider proof systems $\mathcal{P}'$ and $\mathcal{P}''$ defined as follows:

- $\mathcal{P}'$ *is induced by the Hilbert system* $\langle C, R'\rangle$, *where* $C_0 = \{\varphi\}$, $C_1 = \{X, G\}$ *and* R' *contains the rules* $r_k = \langle\{X^{2k}\varphi\}, X^{2k+1}\varphi\rangle$;
- $\mathcal{P}''$ *is the proof system with the same signature* C *and whose derivations are either Hilbert derivations from the single rule* $r_k = \langle\{X^{2k-1}\varphi\}, X^{2k}\varphi\rangle$ *or a derivation* $*$ *proving* $G\varphi$ *from* $\{X^{2n-1}\varphi : n \in \mathbb{N}\}$. *We define* $E \circ d = (E \setminus \{*\}) \circ d$ *whenever* $d \neq *$ *and* $E \circ * = *$. *Note that this is well defined since there are no (non-trivial) derivations of the hypotheses needed for to apply* $*$.

Intuitively, we can see φ *as some property of a system we want to model; in* $\mathcal{P}'$ *we are given that that property holds at time moment* $2k+1$ *if it holds as time moment* $2k$; *in* $\mathcal{P}''$ *we state that it holds at moment* $2k$ *given that it holds at moment* $2k-1$. *The proposition* $G\varphi$ *states that* φ *holds at all odd time instants.*

We now show that, in the fibring $\mathcal{P} = \mathcal{P}' \uplus \mathcal{P}''$, $P_{\{\varphi\}}(d, G\varphi)$ *holds, where the derivation* d *is built in two steps. First we define derivations* s'_{2k} *and* s''_{2k+1} *by*

$$s'_{2k}: \quad \begin{array}{lll} 1 & X^{2k}\varphi & \mathit{Hyp} \\ 2 & X^{2k+1}\varphi & r_k\ 1 \end{array} \qquad\qquad s''_{2k+1}: \quad \begin{array}{lll} 1 & X^{2k+1}\varphi & \mathit{Hyp} \\ 2 & X^{2k+2}\varphi & r_{k+1}\ 1 \end{array}$$

noticing that $P'_{\{x^{2k}\varphi\}}(s'_{2k}, X^{2k+1}\varphi)$ *and* $P''_{\{x^{2k+1}\varphi\}}(s''_{2k+1}, X^{2k+2}\varphi)$ *hold.*

We now define

$$\begin{array}{rcl} d_0 & = & \varphi \text{ (as a sequence)} \\ d_{2k+1} & = & \langle\{d_{2k}\}, s'_{2k}\rangle \\ d_{2k+2} & = & \langle\{d_{2n+1}\}, s''_{2k+1}\rangle \\ d & = & \langle\{d_{2k-1} : k \in \mathbb{N}\}, *\rangle \end{array}$$

and show by induction that $P_{\{\varphi\}}(d_n, X^n\varphi)$ *holds. Indeed:*

- $P'_{\{\varphi\}}(\varphi, \varphi)$ *holds.*
- *Assuming that* $P_{\{\varphi\}}(d_n, X^n\varphi)$ *holds, there are two cases. Suppose that* $n+1 = 2k+1$; *then the thesis holds because* $d = \langle\{d_{2k}\}, s'_{2k}\rangle$, $P_{\{\varphi\}}(d_{2k}, X^{2k}\varphi)$ *holds by induction hypothesis and* $P'_{\{X^{2k}\varphi\}}(s'_{2k}, \varphi X^{2k+1})$ *holds as remarked above. Suppose that* $n+1 = 2k+2$; *then the thesis holds because* $d = \langle\{d_{2k+1}\}, s''_{2k+1}\rangle$, $P_{\{\varphi\}}(d_{2k+1}, X^{2k+1}\varphi)$ *holds by induction hypothesis and* $P''_{\{X^{2k+1}\varphi\}}(s''_{2k+1}, X^{2k+2}\varphi)$ *holds.*

Finally, from $P_{\{\varphi\}}(\{d_{2k-1} : k \in \mathbb{N}\}, \{X^{2k-1}\varphi : k \in \mathbb{N}\})$ *and* $P''_{\{X^{2k-1}\varphi : k\in\mathbb{N}\}}(*, G\varphi)$ *we conclude that* $P_{\{\varphi\}}(d, G\varphi)$. *It is easy to see that no finite number of steps will suffice to derive* $G\varphi$ *from* $\{\varphi\}$. $\triangleleft$

We investigate the relationship between the fibring and the original proof systems showing that the latter are weaker than the former. Also of interest is to analyze how the fibring relates with proof systems that are stronger than the components.

PROPOSITION 3.15
Fibring $\mathcal{P}' \uplus \mathcal{P}''$ of two proof systems is a proof system and moreover

$$\mathcal{P}' \leq \mathcal{P}' \uplus \mathcal{P}'' \text{ and } \mathcal{P}'' \leq \mathcal{P}' \uplus \mathcal{P}''.$$

PROOF. (1) We start by proving that $\mathcal{P}' \uplus \mathcal{P}''$ is a proof system.
(i) Right reflexivity. If $\varphi \in \Gamma$, then $\tau'(\varphi) \in \tau'(\Gamma)$, hence $P'_{\tau'(\Gamma)}(d', \tau'(\varphi))$ holds for some $d' \in D'$. Therefore $P_\Gamma(d', \varphi)$ also holds, so $P_\Gamma(D, \varphi)$ holds.
(ii) Monotonicity. Suppose $\Gamma_1 \subseteq \Gamma_2$ and suppose that $P_{\Gamma_1}(D, \varphi)$ holds. Then there is α such that $P_{\Gamma_1}(D_\alpha, \varphi)$ holds. We show by induction on α that $P_{\Gamma_2}(D_\alpha, \varphi)$ holds. (a) $\alpha = 0$. Without loss of generality, assume that there is $d' \in D'$ such that $P'_{\tau'(\Gamma_1)}(d', \tau'(\varphi))$ holds and, by monotonicity of $\mathcal{P}'$, also $P'_{\tau'(\Gamma_2)}(d', \tau'(\varphi))$ holds, thus $P_{\Gamma_2}(d', \varphi)$ holds and so $P_{\Gamma_2}(D_0, \varphi)$. (b) $\alpha = \beta+1$. Assume $P_{\Gamma_1}(D_{\beta+1}, \varphi)$ holds. Hence there is Ψ such that $P_{\Gamma_1}(D_\beta, \Psi)$ and $P_\Psi(D, \varphi)$. Using the induction hypothesis we have $P_{\Gamma_2}(D_\beta, \Psi)$ and by definition of D we get $P_{\Gamma_2}(D_\alpha, \varphi)$. (c) α is a limit ordinal. This case is simple.
(iii) Compositionality. Immediate by definition of $\circ$.
(iv) Variable exchange. Let ρ be a renaming substitution and suppose that $P_\Gamma(D, \varphi)$ holds. Then there is α such that $P_\Gamma(D_\alpha, \varphi)$ holds. We prove by induction on α that $P_{\rho(\Gamma)}(D_\alpha, \rho(\varphi))$ holds. (a) $\alpha = 0$. Suppose that d is $d' \in D'$; then $P'_{\tau'(\Gamma)}(d', \tau'(\varphi))$ holds. We have to show that there is $e' \in D'$ such that $P'_{\tau'(\rho(\Gamma))}(e', \tau'(\rho(\varphi)))$ holds. Let $\rho' : \Xi \to L(C')$ be the renaming substitution such that $\rho'(\xi) = \tau'(\rho({\tau'}^{-1}(\xi)))$. The variable exchange property for $\mathcal{P}'$ leads to the existence of $e' \in D$ such that $P'_{\rho'(\tau'(\Gamma))}(e', \rho'(\tau'(\varphi)))$ holds. Since $\rho'(\tau'(\psi)) = \tau'(\rho(\psi))$ for every $\psi \in L(C)$ we conclude that $P'_{\tau'(\rho(\Gamma))}(e', \tau'(\rho(\varphi)))$ holds. If d is $d'' \in D''$ the proof is similar. (b) $\alpha = \beta+1$. Since $P_\Gamma(D_{\beta+1}, \varphi)$ then there is Ψ such that $P_\Gamma(D_\beta, \Psi)$ and $P_\Psi(D, \varphi)$. For each $\psi \in \Psi$ there exists $e \in D_\beta$ for which $P_\Gamma(e, \psi)$ holds, and by induction hypothesis, there is some $e'(\psi) \in D_\beta$ for which $P_{\rho(\Gamma)}(e'(\psi), \rho(\psi))$ holds. Thus $P_{\rho(\Gamma)}(D_\beta, \rho(\Psi))$ for $E' = \{e'(\psi) : \psi \in \Psi\}$. Using a reasoning similar to the one for the basis we conclude that $P_{\rho(\Psi)}(D, \rho(\varphi))$, and so $P_{\rho(\Gamma)}(D_\alpha, \rho(\varphi))$ holds. (c) α is a limit ordinal. Straightforward.
(2) It remains to show that $\mathcal{P}' \leq \mathcal{P}' \uplus \mathcal{P}''$ and $\mathcal{P}'' \leq \mathcal{P}' \uplus \mathcal{P}''$. Since both situations are similar, we show the first one, which amounts to showing that $P'_{\Gamma'}(D', \varphi') \leq P_{\Gamma'}(D, \varphi')$ for $\Gamma' \cup \{\varphi'\} \subseteq L(C')$. Suppose that $P'_{\Gamma'}(d', \varphi')$ holds. Then, since τ' is a renaming substitution on $L(C')$, there is a derivation $d \in D'$ such that $P'_{\tau'(\Gamma)}(d, \tau'(\varphi))$ holds, and therefore $P_\Gamma(d, \varphi)$ holds. ∎

We need an auxiliary result before characterizing fibring in the class of proof systems that are stronger than the components.

LEMMA 3.16
Fibring of proof systems closed for substitution is also closed for substitution.

The proof of this result is similar to the one showing that the fibring satisfies variable exchange, and for this reason we omit it.

PROPOSITION 3.17
Fibring is the supremum in the class of proof systems closed for substitution.

Proof. By Propositions 3.15 and 3.16, one only has to prove that $\mathcal{P}' \uplus \mathcal{P}'' \leq \mathcal{P}'''$ whenever $\mathcal{P}' \leq \mathcal{P}'''$ and $\mathcal{P}'' \leq \mathcal{P}'''$. Let $\mathcal{P}'''$ be such a system; we have to show that $P_\Gamma(D,\varphi) \leq P'''_\Gamma(D''',\varphi)$. Assume that $P_\Gamma(D,\varphi)$ holds. Then there is $d \in D$ such that $P_\Gamma(d,\varphi)$. We prove by induction on d that there is d''' such that $P'''_\Gamma(d''',\varphi)$.
(i) Assume that d is $d' \in D'$. Then $P_{\tau'(\Gamma)}(d',\tau'(\varphi))$ holds and so by the hypothesis on $\mathcal{P}'''$ there is $e''' \in D'''$ such that $P'''_{\tau'(\Gamma)}(e''',\tau'(\varphi))$ also holds. Since $\mathcal{P}'''$ is closed for substitution there is $d''' \in D'''$ such that $P'''_{\tau'^{-1}(\tau'(\Gamma))}(d''',\tau^{-1}(\tau'(\varphi)))$ holds and so there is $d''' \in D'''$ such that $P'''_\Gamma(d''',\varphi)$ holds. If d is $d'' \in D''$, the situation is analogous.
(ii) Assume that d is $\langle E, f\rangle$. Then there is $\Psi \subseteq L(C)$ such that $P_\Psi(f,\varphi)$ and $P_\Gamma(E,\Psi)$ hold. By induction hypothesis there is E''' such that $P'''_\Gamma(E''',\Psi)$ holds. Using a reasoning similar to the one above, there is $d''' \in D'''$ such that $P'''_\Psi(d''',\varphi)$ holds. Hence $P'''_\Gamma(E''' \circ d''',\varphi)$ holds. ∎

Now we turn our attention toward preservation of properties by fibring starting with compactness.

Theorem 3.18
The fibring of compact proof systems is compact.

Proof. We prove that there is $\Phi \subseteq \Gamma$ finite such that $P_\Phi(d,\varphi)$ whenever $P_\Gamma(d,\varphi)$ by induction on d.
(i) Let $d \in D'$. Then $P'_{\tau'(\Gamma)}(d,\tau'(\varphi))$, so there are $\Phi' \subseteq \tau'(\Gamma)$ and $d' \in D'$ such that $P'_{\Phi'}(d',\tau'(\varphi))$ and so $P_{\tau'^{-1}(\Phi')}(d',\varphi)$ where $\tau'^{-1}(\Phi') \subseteq \Gamma$ is finite. The case $d \in D''$ is analogous.
(ii) Let $d = \langle E, d'\rangle$. Then there is Ψ such that $P_\Gamma(E,\Psi)$ and $P_\Psi(d',\varphi)$ hold. Using a reasoning similar to the one in the basis, we can conclude that there are $\Phi \subseteq \Psi$ finite and $f \in D$ such that $P_\Phi(f,\varphi)$ holds. On the other hand, since $P_\Gamma(E,\Phi)$ holds, then by induction hypothesis there are $\Omega \subseteq \Gamma$ finite and $F \subseteq D$ such that $P_\Omega(F,\Phi)$, and so $P_\Omega(\langle F, f\rangle,\varphi)$. ∎

As a consequence, the finite-derivation fibring of compact proof systems has the same deductive power as their fibring, i.e. the value of $P_\Gamma(D,\varphi)$ is independent on whether D is obtained by fibring or by finite-derivation fibring.

3.4 Abduction

Preservation of effectiveness in the fibring can be investigated. We make use in our reasoning of the Church-Turing postulate. Before that, we make a detour because we have to assume a further requirement concerning compositionality of derivations. We need to be able to "calculate" a set Ψ in the definition of compositionality. Sometimes we can easily find the set Ψ referred to in the definition of compositionality, e.g. when we are dealing with proof systems induced by known calculi. This motivates the following definition of hypothesis-abductible proof system.

Let $\mathcal{P} = \langle C, D, \circ, P\rangle$ be a proof system. An *abduction function* for $\mathcal{P}$ is a computable function $\mathrm{Abd} : D \to \wp_{\mathrm{fin}}\wp_{\mathrm{fin}}L(C)$ such that for any $\Gamma \subseteq L(C)$, if $P_\Gamma(d,\varphi)$ holds then there is $\Delta \in \mathrm{Abd}(d)$ such that $P_\Delta(d,\varphi)$ and $\Delta \subseteq \Gamma$.

A proof system $\mathcal{P}$ is said to be *hypothesis-abductible* if there is an abduction function for $\mathcal{P}$.

The intuition is as follows: for $d \in D$, $\mathrm{Abd}(d)$ contains sets of hypotheses that make d a valid derivation in proof system $\mathcal{P}$. It is discussible whether this function should also depend on the formula φ one wants to derive. In all examples shown the construction will independent of this formula, supporting the claim that in general that is also the case; however, it would be straightforward to extend our results to abduction functions depending on a derivation and a formula.

PROPOSITION 3.19
Every hypothesis-abductible proof system is compact.

PROOF. Straightforward from the definition. ∎

Hilbert calculi, sequent calculi and tableau systems all induce hypothesis-abductible proof systems.

EXAMPLE 3.20
Let H be a Hilbert calculus and $\mathcal{P}(H)$ the corresponding induced proof system. Then Abd_H is an abduction function for $\mathcal{P}(H)$, where

$$Abd_H(d) = \{\{\varphi : \varphi \textit{ occurs in } d \textit{ justified by } Hyp\}\}.$$

EXAMPLE 3.21
Let G be a sequent calculus and $\mathcal{P}(G)$ the corresponding induced proof system. Then Abd_G is an abduction function for $\mathcal{P}(G)$, where

$$Abd_G(d_{\Gamma \to \Delta}) = \{\Gamma\}.$$

EXAMPLE 3.22
Let S be a tableau calculus and $\mathcal{P}(S)$ the corresponding induced proof system. Then Abd_S is an abduction function for $\mathcal{P}(S)$, where

$$Abd_S(d) = \{\{\Delta\} \textit{ if the first set in } d \textit{ is } (1\!:\!\Delta) \cup (0\!:\!\Phi)\}.$$

Notice that in all these examples there can be spurrious formulas in the only set in $\mathrm{Abd}(d)$; this is because we do not analyze the derivations except in a syntactical way. In other words, the set of hypotheses generated makes d a valid derivation, but nothing forbids that there be another derivation equivalent (in some sense) to d that depends on a smaller set of formulas.

The examples above suggest that $\mathrm{Abd}(d)$ might have been defined to return simply a finite set of formulas. However, the fibring of hypothesis-abductible proof systems would then not be necessarily be hypothesis-abductible. This is illustrated by Example 3.24.

THEOREM 3.23
The finite-derivation fibring of hypothesis-abductible proof systems is hypothesis-abductible.

PROOF. Let $\mathcal{P}'$ and $\mathcal{P}''$ be proof systems with abduction functions Abd' and Abd'' respectively and let $\mathcal{P}$ be their finite-derivation fibring.

Define $\mathrm{Abd}(d)$ by recursion on d as follows.

- If $d \in D' \cap D''$, then $\mathrm{Abd}(d) = {\tau'}^{-1}(\mathrm{Abd}'(d)) \cup {\tau''}^{-1}(\mathrm{Abd}''(d))$.
- If $d \in D'$, then $\mathrm{Abd}(d) = {\tau'}^{-1}(\mathrm{Abd}'(d))$.

- If $d \in D''$, then $\mathrm{Abd}(d) = \tau''^{-1}(\mathrm{Abd}''(d))$.
- If $d = \langle E, d_0 \rangle$, then $\mathrm{Abd}(d)$ contains all sets generated as follows. For each $E' = \{e_1, \ldots, e_n\} \subseteq E$, let $\mathrm{Abd}(e_i) = \{\Psi_{i,1}, \ldots, \Psi_{i,k_i}\}$. Then $\Psi_{1,i_1} \cup \Psi_{2,i_2} \cup \ldots \cup \Psi_{n,i_n} \in \mathrm{Abd}(d)$ for all values of $i_1, \ldots, i_n$ such that the previous expression makes sense.

Notice that this definition is not really recursive, since the apparent recursive call in the last case always reduces to a call of Abd' or Abd''. Also, since E is finite, the resulting set is also finite (though possibly very large).

We now show that Abd thus defined is an abduction function. Suppose that $P_\Gamma(d, \varphi)$ holds. If $d \in D'$ or $d \in D''$ then the result follows because Abd' and Abd'' are both abduction functions. Suppose that $d = \langle E, d_0 \rangle$. Then there is Δ such that $P_\Gamma(E, \Delta)$ and $P_\Delta(d_0, \varphi)$ both hold. By Theorem 3.18 we may assume that Δ is finite.

Let $E' \subseteq E$ be a minimal set such that $P_\Gamma(E', \Delta)$ still holds. Since E' is finite, let $d_1, \ldots, d_n$ be its elements; by minimality $\Delta = \{\varphi_1, \ldots, \varphi_n\}$ with $P_\Gamma(d_i, \varphi_i)$ holding for $i = 1, \ldots, n$. Then for each i there is $\Psi_i \in \mathrm{Abd}(d_i)$ such that $P_{\Psi_i}(d_i, \varphi_i)$ holds and $\Psi_i \subseteq \Gamma$. Take $\Psi = \Psi_1 \cup \ldots \Psi_n \in \mathrm{Abd}(d)$; trivially $\Psi \subseteq \Gamma$; furthermore, $P_\Psi(E', \Delta)$ holds by construction, hence $P_\Psi(E, \Delta)$ also holds, and since we assumed $P_\Delta(d_0, \varphi)$ we conclude that $P_\Psi(d, \varphi)$. ■

We illustrate the abduction function defined above with a simple example that also makes clear the need for the complex type of its output.

Example 3.24

Consider the systems $\mathcal{P}' = \mathcal{P}(H_B)$ introduced in Example 2.4 and $\mathcal{P}'' = \mathcal{P}(G_{\vee,\Rightarrow,\neg})$, induced by the sequent calculus with connectives $\vee$, $\Rightarrow$ and $\neg$ and the rules $R\Rightarrow$, $L\Rightarrow$, $R\neg$ and $L\neg$ from Example 2.6 together with the two following rules for $\vee$.

$$\frac{\Delta_1 \to \xi_1, \xi_2, \Delta_2}{\Delta_1 \to (\xi_1 \vee \xi_2), \Delta_2}\ R\vee \qquad \frac{\Delta_1, \xi_1 \to \Delta_2 \quad \Delta_1, \xi_2 \to \Delta_2}{\Delta_1, (\xi_1 \vee \xi_2) \to \Delta_2}\ L\vee$$

In $\mathcal{P}'$, we have the following derivation d_1.

1	$\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$	*Ax*
2	ξ_1	*Hyp*
3	$\xi_2 \Rightarrow \xi_1$	*MP 1,2*

According to the definition above, $Abd_{\mathcal{P}'}(d_1) = \{\{\xi_1\}\}$.

In the same system, we can also build d_2 as follows.

1	$(\neg\xi_2) \Rightarrow ((\neg\xi_1) \Rightarrow (\neg\xi_2))$	*Ax*
2	$(\neg\xi_2)$	*Hyp*
3	$(\neg\xi_1) \Rightarrow (\neg\xi_2)$	*MP 1,2*
4	$((\neg\xi_1) \Rightarrow (\neg\xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1)$	*Ax*
5	$\xi_2 \Rightarrow \xi_1$	*MP 4,3*

In this case, $Abd_{\mathcal{P}'}(d_2) = \{\{\neg\xi_2\}\}$.

Turning now to $\mathcal{P}''$, we can build the following derivation d.

1	$\xi_2 \Rightarrow \xi_1 \to \xi_1 \vee (\neg\xi_2)$	*R∨ 2*
2	$\xi_2 \Rightarrow \xi_1 \to \xi_1, (\neg\xi_2)$	*L⇒ 3,4*
3	$\to \xi_2, \xi_1, (\neg\xi_2)$	*R¬ 5*
4	$\xi_1 \to \xi_1, (\neg\xi_2)$	*Ax*
5	$\xi_2 \to \xi_2, \xi_1$	*Ax*

By the definition above, $Abd_{\mathcal{P}''}(d) = \{\{\xi_2 \Rightarrow \xi_1\}\}$.

Consider now the fibring $\mathcal{P} = \mathcal{P}' \uplus \mathcal{P}''$ *and the derivation* $d^* = \langle\{d_1, d_2\}, d\rangle$ *in this system. The construction in the proof of Theorem 3.23 yields*

$$Abd_{\mathcal{P}}(d^*) = \{\emptyset, \{\xi_1\}, \{\neg\xi_2\}, \{\xi_1, \neg\xi_2\}\}.$$

It is easy to see that $P_\Gamma(d^*, \varphi)$ *holds iff* φ *coincides with* $\xi_1 \vee (\neg\xi_2)$ *and* Γ *contains either* ξ_1 *or* $\neg\xi_2$.

THEOREM 3.25
Let $\mathcal{P}'$ *and* $\mathcal{P}''$ *be hypothesis-abductible and decidable proof systems. Then their fibring is a decidable proof system.*

PROOF. Assume that $\mathcal{P}'$ and $\mathcal{P}''$ are decidable proof systems and let Abd' and Abd'' be abduction functions for them. Let Abd be an abduction function for $\mathcal{P} = \mathcal{P}' \uplus \mathcal{P}''$ (in particular, Abd may be the function defined in the proof of Theorem 3.23). The following recursive algorithm allows us to decide whether $P_\Gamma(d, \varphi)$ holds.

1. If $d \in D'$ and $P_{\tau'(\Gamma)}(d, \tau'(\varphi))$ holds, then output 1.
2. If $d \in D''$ and $P_{\tau''(\Gamma)}(d, \tau''(\varphi))$ holds, then output 1.
3. If d is not of the form $\langle E, d_0\rangle$ then output 0.
4. Let $d = \langle E, d_0\rangle$ and let $\mathrm{Abd}_d = \{\Psi_1, \ldots, \Psi_n\}$.
5. For $i = 1, \ldots, n$ do
   (a) If $P_{\Psi_i}(d_0, \varphi)$ holds and $P_\Gamma(E, \Psi_i)$ holds, output 1.
   (b) Otherwise, increment i.
6. Return 0.

Termination of the algorithm follows from the fact that the abduction function only returns a finite number of sets and that P', P'' and the recursive call to P always terminate.

For correctness, suppose first that $P_\Gamma(d, \varphi)$ holds. If $d \in D'$ or $d \in D''$, then either the first or the second steps will return answer 1. Otherwise, let $d = \langle E, d_0\rangle$ and suppose that $P_\Gamma(E, \Psi)$ and $P_\Psi(d, \varphi)$ hold. By definition of abduction, $P_\Delta(d, \varphi)$ also holds for some $\Delta \in \mathrm{Abd}(d)$ with $\Delta \subseteq \Psi$; this guarantees that Δ will be one of the Ψ_is in the cycle of the previous algorithm and that the check in the first step of the cycle will succeed (using the induction hypothesis). Hence the algorithm always returns 1 in this case.

Conversely, suppost that the algorithm returns 1 to the question $P_\Gamma(d, \varphi)$. If $d \in D'$ or $d \in D''$, then one of the first two steps must have been used, so $P_\Gamma(d, \varphi)$ holds. Otherwise, d must be of the form $\langle E, d_0\rangle$ and there is some $\Psi_i \in \mathrm{Abd}(d_0)$ such that $P_{\Psi_i}(d_0, \varphi)$ and $P_\Gamma(E, \Psi_i)$ both hold; but this implies that $P_\Gamma(d, \varphi)$ holds. ∎

For illustration, we show how this algorithm works in the previous example.

EXAMPLE 3.26
Consider again the proof system $\mathcal{P}$ *presented in the previous example. For simplicity we omit the translation steps (which would change only variable indices, irrelevant since proof systems are closed under renaming substitutions). To decide whether* $P_\Gamma(d^*, \xi_1 \vee (\neg\xi_2))$ *holds, the previous algorithm will first compute* $Abd_{\mathcal{P}''}(d) = \{\{(\xi_2 \Rightarrow \xi_1)\}\}$, *then check whether* $P_{\{(\xi_2 \Rightarrow \xi_1)\}}(d, \xi_1 \vee (\neg\xi_2))$ *holds (it does), and finally check whether* $P_\Gamma(\{d_1, d_2\}, (\xi_2 \Rightarrow \xi_1))$ *holds.*

We now return briefly to the question of whether recursiveness of the set of theorems is preserved by fibring. Notice that, in general, a derivation with no hypotheses in the fibring of two proof systems requires derivations *with* hypotheses in the original systems, therefore an affirmative answer is unlikely. Even if the systems are finitary and some version of the Deduction Theorem is available, some form of abduction will be needed to find the intermediary hypotheses, and since (unlike in the previous case) no derivation is available to start with, it is not at all obvious how such an abduction mechanism would work. For this reason, we conjecture that recursiveness of theoremhood is not preserved by fibring, and do not address that question further in the present paper.

3.5 Proof systems vs consequence systems

The subsection is dedicated to the investigation of the relationship between proof systems and consequence systems. We start by discussing the generation of a consequence system out of a proof system.

PROPOSITION 3.27
A proof system $\mathcal{P} = \langle C, D, \circ, P\rangle$ induces a consequence system $\mathcal{C}(\mathcal{P}) = \langle C, \vdash\rangle$ where $\Gamma^{\vdash} = \{\varphi \in L(C) : P_\Gamma(D, \varphi)\}$.

PROOF. (i) Extensivity. Follows directly from the right reflexivity of P_Γ. (ii) Monotonicity. Suppose that $\Gamma_1 \subseteq \Gamma_2$ and that $\varphi \in \Gamma_1$. Then $P_{\Gamma_1}(D, \varphi)$ holds. By the monotonicity of $\mathcal{P}$ we have $P_{\Gamma_1}(D, \varphi) \leq P_{\Gamma_2}(D, \varphi)$ hence $P_{\Gamma_2}(D, \varphi)$ holds and so $\varphi \in \Gamma_2^{\vdash}$. (iii) Idempotence. Suppose that $\varphi \in (\Gamma^{\vdash})^{\vdash}$. Then there is $d \in D$ such that $P_{\Gamma^{\vdash}}(d, \varphi)$. On the other hand, there is $E \subseteq D$ such that $P_\Gamma(E, \Gamma^{\vdash})$. Hence by compositionality in $\mathcal{P}$ we have $P_\Gamma(E \circ d, \varphi)$ and so $\varphi \in \Gamma^{\vdash}$. (iv) Closure for renaming substitution. Assume that ρ is a renaming substitution and $\varphi \in \Gamma^{\vdash}$. Then $P_\Gamma(D, \varphi)$ holds and so, by variable exchange in $\mathcal{P}$, $P_{\rho(\Gamma)}(D, \rho(\varphi))$ holds and so $\rho(\varphi) \in \rho(\Gamma)^{\vdash}$. ∎

We will now investigate how properties of the proof system are propagated to the induced consequence system. It is very simple (similar to closure for renaming substitution in the proof of the result above) to prove the following result:

PROPOSITION 3.28
If $\mathcal{P}$ is closed for substitution, then so is $\mathcal{C}(\mathcal{P})$.

It is worthwhile to detail compactness, effectiveness and strong effectiveness.

PROPOSITION 3.29
If $\mathcal{P}$ is compact, then so is $\mathcal{C}(\mathcal{P})$.

PROOF. Suppose that $\mathcal{P}$ is compact and $\varphi \in \Gamma^{\vdash}$. Then $P_\Gamma(D, \varphi)$ holds and so by, compactness of $\mathcal{P}$, there is $\Phi \subseteq \Gamma$ finite such that $P_\Phi(D, \varphi)$ also holds and so $\varphi \in \Phi^{\vdash}$. ∎

The following result is very easy to prove.

PROPOSITION 3.30
If $\mathcal{P}$ is an effective proof system, then $\mathcal{C}(\mathcal{P})$ is semi-decidable.

Proof. Assume that $\mathcal{P}$ is an effective proof system and $\Gamma \subseteq L(C)$ is a recursive set. Then $\varphi \in \Gamma^{\vdash}$ iff there is $d \in D$ such that $P_\Gamma(d, \varphi)$ where P_Γ is a recursive relation. By the projection theorem, $\Gamma^{\vdash}$ is a recursively enumerable set. ∎

Proposition 3.31
The consequence system $\mathcal{C}(\mathcal{P})$ is strongly semi-decidable when $\mathcal{P}$ is a strongly effective proof system.

Proof. Assume that $\mathcal{P}$ is a strongly effective proof system. Let $\Gamma \subseteq L(C)$ be a recursively enumerable set. Then P_Γ is recursively enumerable, so either $P_\Gamma = \emptyset$ or there is a partial recursive function $f : \mathbb{N} \to D \times L(C)$ such that $f(\mathbb{N}) = P_\Gamma$. If $P_\Gamma = \emptyset$ then $\Gamma^{\vdash} = \emptyset$, which is a recursively enumerable set (in this case Γ itself must be empty). Otherwise, define $f^* : \mathbb{N} \to L(C)$ such that $f^*(n)$ is the second component of $f(n)$. Clearly f^* is recursive. Furthermore, $\varphi \in \Gamma^{\vdash}$ iff $P_\Gamma(d, \varphi)$ holds for some $d \in D$ iff $f(n) = \langle d, \varphi\rangle$ for some n iff $f^*(n) = \varphi$. Hence the co-domain of f^* is precisely $\Gamma^{\vdash}$ and $\Gamma^{\vdash}$ is recursively enumerable. ∎

Also of interest is the relationship between a calculus and the proof system it induces. We do a parametric proof of the following result.

Proposition 3.32
Let Calc be a (Hilbert, sequent, tableau) calculus. Then $\mathcal{C}(\mathcal{P}(\mathit{Calc})) = \mathcal{C}(\mathit{Calc})$.

Proof. Since both $\mathcal{C}(\mathcal{P}(\text{Calc}))$ and $\mathcal{C}(\text{Calc})$ share the same signature C, all one needs to show is that the closure of a set $\Gamma \subseteq L(C)$ is the same in both cases. Let $\mathcal{C}(\mathcal{P}(\text{Calc})) = \langle C, \vdash_1\rangle$ and $\mathcal{C}(\text{Calc}) = \langle C, \vdash_2\rangle$. Then $\varphi \in \Gamma^{\vdash_1}$ iff $P_\Gamma(D, \varphi)$ holds in $\mathcal{P}(H)$ iff there is a Calc-derivation of φ from Γ in D iff $\varphi \in \Gamma^{\vdash_2}$. ∎

The following result indicates that relationships between proof systems are preserved by the induced consequence systems.

Proposition 3.33
Let P and P' be proof systems such that $P \leq P'$. Then $\mathcal{C}(P) \leq \mathcal{C}(P')$.

Proof. Let $\mathcal{P} \leq \mathcal{P}'$ and $\Gamma \subseteq L(C)$. Then $C \subseteq C'$; suppose that $\varphi \in \Gamma^{\vdash}$. Then $P_\Gamma(D, \varphi)$ holds, whence $P'_\Gamma(D', \varphi)$ also holds since $\mathcal{P} \leq \mathcal{P}'$ and so $\varphi \in \Gamma^{\vdash'}$. ∎

As a special case we conclude $\mathcal{C}(\mathcal{P}') \leq \mathcal{C}(\mathcal{P}' \uplus \mathcal{P}'')$ and $\mathcal{C}(\mathcal{P}'') \leq \mathcal{C}(\mathcal{P}' \uplus \mathcal{P}'')$.

Now we show how to generate a proof system out of a consequence system.

Proposition 3.34
A consequence system $\mathcal{C}$ induces a proof system $\mathcal{P}(\mathcal{C})$ with the same signature as follows: $D = \{*\}$; $E \circ * = *$; $P_\Gamma(*, \varphi)$ holds iff $\varphi \in \Gamma^{\vdash}$.

Proof. (i) Right reflexivity. Since $\vdash$ is extensive, $\Gamma \subseteq \Gamma^{\vdash}$ for every $\Gamma \subseteq L(C)$, so $P_\Gamma(D, \Gamma)$ holds. (ii) Monotonicity. Assume that $\Gamma_1 \subseteq \Gamma_2$ and $P_{\Gamma_1}(D, \varphi)$ holds. Then $\varphi \in \Gamma_1^{\vdash}$; by monotonicity of $\mathcal{C}$, $\Gamma_1^{\vdash} \subseteq \Gamma_2^{\vdash}$, hence $\varphi \in \Gamma_2^{\vdash}$, and so $P_{\Gamma_2}(D, \varphi)$. (iii) Idempotence. Suppose that $P_\Gamma(E, \Psi)$ and $P_\Psi(d, \varphi)$ hold. Then $\Psi \subseteq \Gamma^{\vdash}$ and $\varphi \in \Psi^{\vdash}$, hence, by monotonicity of $\mathcal{C}$, $\varphi \in (\Gamma^{\vdash})^{\vdash}$ and so, by idempotence of $\vdash$, $\varphi \in \Gamma^{\vdash}$. Therefore $P_\Gamma(E \circ d, \varphi)$ holds. (iv) Variable exchange. Assume that ρ is a renaming substitution and that that $P_\Gamma(D, \varphi)$ holds. Then $\varphi \in \Gamma^{\vdash}$, hence $\rho(\varphi) \in \rho(\Gamma)^{\vdash}$ and so $P_{\rho(\Gamma)}(D, \rho(\varphi))$. ∎

We can show easily that the induced proof system is closed for substitution and compact whenever the consequence system has the same properties. A proof system $\mathcal{P}$ can be compared with the proof system generated by the consequence system induced by $\mathcal{P}$ as the following result states.

Proposition 3.35
For any proof system $\mathcal{P}$, $\mathcal{P} \leq \mathcal{P}(\mathcal{C}(\mathcal{P}))$.

Proof. Straightforward. Since in both constructions the signature does not change, all that is left to show is that, if $P_\Gamma(D,\varphi)$ holds in $\mathcal{P}$, then $P_\Gamma(*,\varphi)$ holds in $\mathcal{P}(\mathcal{C}(\mathcal{P}))$. Assume that $P_\Gamma(D,\varphi)$ holds in $\mathcal{P}$. Then $\varphi \in \Gamma^{\vdash}$ and so $P_\Gamma(\{*\},\varphi)$ holds in $\mathcal{P}(\mathcal{C}(\mathcal{P}))$. ∎

The opposite relation also holds: for every consequence system: $\mathcal{C} = \mathcal{C}(\mathcal{P}(\mathcal{C}))$.

Finally, we relate the consequence system induced by the fibring of proof systems with the fibring of the consequence systems induced by the proof systems.

Proposition 3.36
The fibring of proof systems has the following property.

$$\mathcal{C}(\mathcal{P}' \uplus \mathcal{P}'') = \mathcal{C}(\mathcal{P}') \uplus \mathcal{C}(\mathcal{P}'')$$

Proof. The signature of both $\mathcal{C}(\mathcal{P}' \uplus \mathcal{P}'')$ and $\mathcal{C}(\mathcal{P}') \uplus \mathcal{C}(\mathcal{P}'')$ is $C = C' \cup C''$. Denoting $\mathcal{C}(\mathcal{P}' \uplus \mathcal{P}'')$ by $\langle C, \vdash_a \rangle$ and $\mathcal{C}(\mathcal{P}') \uplus \mathcal{C}(\mathcal{P}'')$ by $\langle C, \vdash_b \rangle$, all that is left to show is that $\Gamma^{\vdash_a} = \Gamma^{\vdash_b}$ for all $\Gamma \subseteq L(C)$.
(i) We start by showing that $\Gamma^{\vdash_a} \subseteq \Gamma^{\vdash_b}$. Suppose $\varphi \in \Gamma^{\vdash_a}$. Then $P_\Gamma(D,\varphi)$ holds, hence $P_\Gamma(d,\varphi)$ holds for some $d \in D$. We prove that $\varphi \in \Gamma^{\vdash_b}$ by induction on d. (a) If d is $d' \in D'$, then $P'_{\tau'(\Gamma)}(d', \tau'(\varphi))$, hence $\tau'(\varphi) \in \tau'(\Gamma)^{\vdash'}$ and therefore $\varphi \in \Gamma^{\vdash_b}$ by definition of fibring of consequence systems. The case where d is $d'' \in D''$ is analogous. (b) If d is $\langle E, d''' \rangle$ with $E \cup \{d'''\} \subseteq D$, then there is a set Ψ such that $P_\Gamma(E,\Psi)$ and $P_\Psi(d''',\varphi)$ both hold, that is, $\Psi \subseteq \Gamma^{\vdash_a}$ and $\varphi \in \Psi^{\vdash_a}$. By induction hypothesis, $\Psi \subseteq \Gamma^{\vdash_b}$ and $\varphi \in \Psi^{\vdash_b}$ and, by idempotence of $\vdash_b$, it follows that $\varphi \in (\Gamma^{\vdash_b})^{\vdash_b} \subseteq \Gamma^{\vdash_b}$.
(ii) Now we show that $\Gamma^{\vdash_b} \subseteq \Gamma^{\vdash_a}$. Suppose now that $\varphi \in \Gamma^{\vdash_b}$. Then $\varphi \in \Gamma^{\vdash_\beta}$ for some ordinal β in the fixed point construction of Proposition 2.13. We prove that $\varphi \in \Gamma^{\vdash_a}$ by induction on β. (a) $\beta = 0$. Straightforward, since $\Gamma \subseteq \Gamma^{\vdash_a}$. (b) If $\varphi \in \Gamma^{\vdash_{\beta+1}}$, then either $\varphi \in {\tau'}^{-1}(\tau'(\Gamma^{\vdash_\beta})^{\vdash'})$ or $\varphi \in {\tau''}^{-1}(\tau''(\Gamma^{\vdash_\beta})^{\vdash''})$; both cases are similar, so assume the first one holds. By induction hypothesis $\Gamma^{\vdash_\beta} \subseteq \Gamma^{\vdash_a}$, so $P_\Gamma(D, \Gamma^{\vdash_\beta})$ holds. Also, from $\varphi \in {\tau'}^{-1}(\tau'(\Gamma^{\vdash_\beta})^{\vdash'})$ we conclude that $\tau'(\varphi) \in \tau'(\Gamma^{\vdash_\beta})^{\vdash'}$, so $P'_{\tau'(\Gamma^{\vdash_\beta})}(d', \tau'(\varphi))$ holds for some $d' \in D'$. Therefore, $P_{\Gamma^{\vdash_\beta}}(d',\varphi)$ also holds and hence $P_\Gamma(D \circ d', \varphi)$ holds, which means that $\varphi \in \Gamma^{\vdash_a}$. (c) β is a limit ordinal: straightforward. ∎

4 Conclusions

In this paper we addressed the problem of heterogenous fibring of logics. In the first place, we studied the well-known notion of consequence system, showed how several kinds of presentations of logics define consequence systems and defined fibring of consequence systems. As a consequence, we showed how we could combine a logic presented syntactically with another presented semantically. We showed that this combination is conservative assuming the original logics are closed under substitution, as well as several results on semi-decidability.

However, this solution is unsatisfactory because no trace is kept of proofs that may exist in the original calculi. For this reason, we introduced the notion of abstract proof system, which intends to abstract the essential properties of logics presented syntactically via some notion of derivation. In particular, this covers Hilbert calculi, sequent calculi and tableau calculi. We showed how to define fibring in this context, and gave some examples illustrating this construction and its advantages over the analogous one obtained by regarding the logics as consequence systems. Finally we defined the property of hypothesis-abduction for an abstract proof system and showed how it could be used to prove preservation of semi-decidability by fibring of proof systems.

Finally we showed how every proof system can be seen as a consequence system and vice-versa, so that even semantically presented logics can be seen to induce abstract proof systems (albeit not-so-interesting ones), and showed that fibring commutes with these views for all the concrete calculi considered.

4.1 Future Work

There are some issues raised in this paper that were not fully explored for lack of space.

First-order case Throughout this paper we only considered propositional signatures for two reasons: they are clearly easier to combine in a natural way, and they suffice for many practical applications. Generalizing the notion of proof system to encompass first- (and higher-) order logics presents different challenges already at that level. Furthermore, since derivations in first-order logic already require the use of rules with provisos, the mechanism of combination itself has to be revised.

Theoremhood As explained at the end of Section 3.4, the question of whether recursiveness of theoremhood is preserved by fibring is far from trivial. Deciding this question will likely require stating the Deduction Theorem in general for proof systems and analyzing how it behaves through fibring, as well as some more work on abduction of hypotheses.

Structural properties of derivations Since we work with proof systems induced from specific calculi, it would be worthwhile to explore properties of derivations that can be abstracted to families of proof systems, and whether they are preserved through fibring. Examples of such properties are the relationship of the size of a formula to the size of its derivation, or invertibility of rules in the case of derivations produced from rules. Another example, with obvious implications in the question discussed in the previous paragraph, is whether a derivation can be generated simply from the structure of a given formula.

Complexity The abduction algorithm for the fibring of two hypothesis-abductible proof systems is very simple and, as such, terribly inefficient. It would be interesting to examine how efficient it can be made and whether good bounds on the time complexity of abduction in the fibring can be proved, based on bounds on the time complexity of the abduction in the components. At the very least, it would be good to know that fibring preserves polynomial-time abduction.

Acknowledgments

The work was supported by Fundação para a Ciência e Tecnologia (FCT) and EU FEDER, namely via the Projects FibLog POCTI / MAT / 37239 / 2001 and Quantlog POCI / MAT / 55796 / 2004 at CLC (Center for Logic and Computation) of IST. Luís Cruz-Filipe was also supported by FCT through postdoc grant SFRH / BPD / 16372 / 2004 and the European Project IST-2005-16004. The authors want to thank the anonymous referee for many useful suggestions.

References

[1] B. Beckert and D. Gabbay. Fibring semantic tableaux. In *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 1397 of *Lecture Notes in Computer Science*, pages 77–92. Springer, 1998.

[2] J. Bell and M. Machover. *A Course in Mathematical Logic*. North-Holland, 1977.

[3] P. Blackburn and M. De Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.

[4] C. Caleiro, W. A. Carnielli, M. E. Coniglio, A. Sernadas, and C. Sernadas. Fibring non-truth-functional logics: Completeness preservation. *Journal of Logic, Language and Information*, 12(2):183–211, 2003.

[5] W. Carnielli, J. Rasga and C. Sernadas. Preservation of interpolation features by fibring. *Journal of Logic and Computation* (in print).

[6] A. Church and S. C. Kleene. Formal definitions in the theory of ordinal numbers. *Fundamenta Mathematica*, 28:11–21, 1937.

[7] M. E. Coniglio, A. Sernadas, and C. Sernadas. Fibring logics with topos semantics. *Journal of Logic and Computation*, 13(4):595–624, 2003.

[8] M. D'Agostino and D. Gabbay. Fibred tableaux for multi-implication logics. In *Theorem Proving with Analytic Tableaux and Related Methods*, volume 1071 of *Lecture Notes in Computer Science*, pages 16–35. Springer, 1996.

[9] M. Finger and D. Gabbay. Combining temporal logic systems. *Notre Dame Journal of Formal Logic*, 37(2):204–232, 1996.

[10] M. Finger and M. Weiss. The unrestricted combination of temporal logic systems. *Logic Journal of the IGPL*, 10(2):165–189, 2002.

[11] D. Gabbay. Fibred semantics and the weaving of logics: Part 1. *Journal of Symbolic Logic*, 61(4):1057–1120, 1996.

[12] D. Gabbay. *Fibring Logics*. Oxford University Press, 1999.

[13] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 2003.

[14] D. Gabbay and V. Shehtman. Products of modal logics. I. *Logic Journal of the IGPL*, 6(1):73–146, 1998.

[15] D. Gabbay and V. Shehtman. Products of modal logics. II. Relativised quantifiers in classical logic. *Logic Journal of the IGPL*, 8(2):165–210, 2000.

[16] D. Gabbay and V. Shehtman. Products of modal logics. III. Products of modal and temporal logics. *Studia Logica*, 72(2):157–183, 2002.

[17] G. Governatori, V. Padmanabhan, and A. Sattar. On fibring semantics for BDI logics. In S. Flesca and G. Ianni, editors, *JELIA*, volume 2424 of *Lecture Notes in Artificial Intelligence*, pages 198–210. Springer Verlag, 2002.

[18] R. Kontchakov, C. Lutz, F. Wolter, and M. Zakharyaschev. Temporalising tableaux. *Studia Logica*, 76(1):91–134, 2004.

[19] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56(4):1469–1485, 1991.

[20] J. Rasga, A. Sernadas, C. Sernadas, and L. Viganò. Fibring labelled deduction systems. *Journal of Logic and Computation*, 12(3):443–473, 2002.

[21] A. Sernadas and C. Sernadas. Combining logic systems: Why, how, what for? *CIM Bulletin*, 15:9–14, December 2003.

[22] A. Sernadas, C. Sernadas, and A. Zanardo. Fibring modal first-order logics: Completeness preservation. *Logic Journal of the IGPL*, 10(4):413–451, 2002.

[23] C. Sernadas, J. Rasga, and W. A. Carnielli. Modulated fibring and the collapsing problem. *Journal of Symbolic Logic*, 67(4):1541–1569, 2002.

[24] R. H. Thomason. Combinations of tense and modality. In *Handbook of Philosophical Logic, Vol. II*, volume 165 of *Synthese Library*, pages 135–165. Reidel, 1984.

[25] A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, volume 43 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, second edition, 2000.

[26] F. Wolter. Fusions of modal logics revisited. In *Advances in Modal Logic, Vol. 1*, volume 87 of *CSLI Lecture Notes*, pages 361–379. Stanford University, 1998.

[27] F. Wolter and M. Zakharyaschev. Temporalizing description logics. In *Frontiers of Combining Systems, 2*, volume 7, pages 379–401. Research Studies Press, 2000.

[28] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. *Journal of Symbolic Logic*, 66(1):414–439, 2001.

Received