The essence of proofs when fibring sequent calculi

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Motivation

- no work on fibring of sequent calculus
- "intuitive" definition not very satisfactory...
- ideas from work on proof systems

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Signatures

Definition

A (propositional) *signature* C is a family of sets indexed by the natural numbers.

The elements of each C_k are called *constructors* or *connectives* of arity k.

We say that $C \subseteq C'$ if $C_k \subseteq C'_k$ for every $k \in \mathbb{N}$.

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Formulas

Definition

Let C be a signature and $\Xi = \{\xi_n : n \in \mathbb{N}\}$ be a countable set of meta-variables.

The *language* $L(C, \Xi)$ is the free algebra over C generated by Ξ.

The elements of $L(C, \Xi)$ are called *formulas*.

Substitutions

Definition

A *substitution* is a map $\sigma : \Xi \to L(C)$.

Substitutions can be inductively extended to formulas and to sets of formulas:

- $\sigma(\gamma)$ is the formula where each $\xi \in \Xi$ is replaced by $\sigma(\xi)$;
- $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$.

In particular, when $\sigma(\xi_n) \in \Xi$ for every n, we say that σ is a *renaming of variables*.

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Definition

A *sequent calculus (given by rules)* is a pair $\mathcal{R} = \langle C, R \rangle$, where C is a signature and R is a set of rules including structural rules and specific rules (for the connectives).

Structural rules

These are chosen among the following.

$$\frac{\xi_1, \Delta_1 \longrightarrow \Delta_2 \quad \Delta_1 \longrightarrow \Delta_2, \xi_1}{\Delta_1 \longrightarrow \Delta_2} \text{ Cut}$$

$$\frac{\Delta_1 \longrightarrow \Delta_2}{\xi_1, \Delta_1 \longrightarrow \Delta_2} \text{ LW} \qquad \frac{\Delta_1 \longrightarrow \Delta_2}{\Delta_1 \longrightarrow \Delta_2, \xi_1} \text{ RW}$$

$$\frac{\Delta_1, \xi_1, \xi_1 \longrightarrow \Delta_2}{\Delta_1, \xi_1 \longrightarrow \Delta_2} \text{ LC} \qquad \frac{\Delta_1 \longrightarrow \xi_1, \xi_1, \Delta_2}{\Delta_1 \longrightarrow \xi_1, \Delta_2} \text{ RC}$$

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Rules for the connectives

These can include:

- Left rules: the antecedent of the conclusion includes a formula $c(\varphi_1, \dots, \varphi_n)$ for some n-ary connective c.
- Right rules: the consequent of the conclusion includes a formula $c(\varphi_1, \dots, \varphi_n)$ for some n-ary connective c.

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Derivations

Definition

A *(rule-)derivation* of a sequent s from a set of sequents Δ in sequent calculus $\mathcal{R}$ is a finite sequence $\{\Delta_{1,i} \longrightarrow \Delta_{2,i}\}_{i=1}^{n}$ of sequents such that:

- $\Delta_{1,1} \longrightarrow \Delta_{2,1}$ is s;
- for each $i = 1, \ldots, n$, one of the following holds:
  - $\Delta_{1,i} \longrightarrow \Delta_{2,i}$ is an axiom (justified by Ax);
  - $\Delta_{1,i} \longrightarrow \Delta_{2,i} \in \Delta$ (justified by Hyp);
  - for some rule $r = \langle\{\theta_1, \ldots, \theta_k\}, \eta\rangle$ and substitution σ, $\Delta_{1,i} \longrightarrow \Delta_{2,i} = \sigma(\eta)$ and $\sigma(\theta_j) \in \{\Delta_{1,i'} \longrightarrow \Delta_{2,i'}\}_{i'=i+1}^{n}$ (justified by $r, i_1, \ldots, i_k$)

Notation: $\Delta \vdash_{\mathcal{R}} s$ or (when Δ is empty) $\vdash_{\mathcal{R}} s$.

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Example: $S4$

All structural rules plus:

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \to \xi_2), \Gamma \longrightarrow \Delta}\ \mathsf{L}\to \qquad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \to \xi_2)}\ \mathsf{R}\to$$

$$\frac{\xi_1, \Gamma_1 \longrightarrow \Diamond(\Delta_1)}{(\Diamond\xi_1), \Box(\Gamma_1), \Gamma_2 \longrightarrow \Delta_2, \Diamond(\Delta_1)}\ \mathsf{L}\Diamond \qquad \frac{\Gamma, \xi_1, (\Box\xi_1) \longrightarrow \Delta}{\Gamma, (\Box\xi_1) \longrightarrow \Delta}\ \mathsf{L}\Box$$

$$\frac{\Box\Gamma_1 \longrightarrow \xi_1, \Delta_1}{\Gamma_2, \Box(\Gamma_1) \longrightarrow (\Box\xi_1), \Diamond(\Delta_1), \Delta_2}\ \mathsf{R}\Box \qquad \frac{\Gamma \longrightarrow \Delta, \xi_1, (\Diamond\xi_1)}{\Gamma \longrightarrow \Delta, (\Diamond\xi_1)}\ \mathsf{R}\Diamond$$

where $\Box(\Gamma) = \{(\Box\varphi) : \varphi \in \Gamma\}$ and $\Diamond(\Gamma) = \{(\Diamond\varphi) : \varphi \in \Gamma\}$

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Derivation in $S4$

Example

The following shows that $\vdash_{S4} \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1)))$.

1.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1)))$	$R\Diamond, 2$
2.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\xi_1 \to (\Box\xi_1))$	$R \to, 3$
3.	$\xi_1 \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\Box\xi_1)$	$R\Box, 4$
4.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), \xi_1$	$R\Diamond, 5$
5.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\xi_1 \to (\Box\xi_1)), \xi_1$	$R \to, 6$
6.	$\xi_1 \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\Box\xi_1), \xi_1$	Ax

Example: D

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$$\frac{\Gamma \longrightarrow \Delta, \xi_1}{\Gamma, (\neg \xi_1) \longrightarrow \Delta} \ \mathsf{L}\neg \qquad \frac{\Gamma, \xi_1 \longrightarrow \Delta}{\Gamma \longrightarrow (\neg \xi_1), \Delta} \ \mathsf{R}\neg$$

$$\frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Box \xi_1)} \ \mathsf{R}\Box \qquad \frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Diamond \xi_1)} \ \mathsf{R}\Diamond$$

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Derivation in D

Example

The following shows that $\longrightarrow \xi_2 \vdash_D \longrightarrow (\Diamond(\xi_1 \to \xi_2))$

1.	$\longrightarrow (\Diamond(\xi_1 \to \xi_2))$	Cut, 2, 5
2.	$(\Box\xi_2) \longrightarrow (\Diamond(\xi_1 \to \xi_2))$	R$\Diamond$, 3
3.	$\xi_2 \longrightarrow (\xi_1 \to \xi_2)$	R $\to$, 4
4.	$\xi_2, \xi_1 \longrightarrow \xi_2$	Ax
5.	$\longrightarrow (\Diamond(\xi_1 \to \xi_2)), (\Box\xi_2)$	RW, 6
6.	$\longrightarrow (\Box\xi_2)$	R$\Box$, 7
7.	$\longrightarrow \xi_2$	Hyp

Definition

Let $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ be sequent calculi.

The (rule-)*fibring* $\mathcal{R}' \uplus \mathcal{R}''$ of $\mathcal{R}'$ and $\mathcal{R}''$ is the sequent calculus $\langle C' \cup C'', R' \cup R'' \rangle$.

Example

We can show that $\vdash_{S4 \uplus D} \longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$

1.	$\longrightarrow \Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	Cut, 2, 5
2.	$(\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$	$R\Diamond''$, 3
3.	$(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))) \longrightarrow (\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$R \rightarrow$, 4
4.	$\xi_2, (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))) \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	Ax
5.	$\longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	RW, 6
6.	$\longrightarrow (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$R\Box''$, 7
7.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	$R\Diamond'$, 8
8.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1))$	$R \rightarrow$, 9
9.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1)$	$R\Box'$, 10
10.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), \xi_1$	$R\Diamond'$, 11
11.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1)), \xi_1$	$R \rightarrow$, 12
12.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1), \xi_1$	Ax

Inspiration

Definition

A *proof system* is a tuple $\mathcal{P} = \langle C, D, \circ, P \rangle$ where C is a signature, D is a set, $\circ : \wp(D) \times D \to D$ and $P = \{P_\Gamma\}_{\Gamma \subseteq L(C)}$ is a family of relations $P_\Gamma \subseteq D \times L(C)$ satisfying the following properties.

- Right reflexivity: if $\gamma \in \Gamma$, then $P_\Gamma(d, \gamma)$ for some $d \in D$;
- Monotonicity: if $\Gamma_1 \subseteq \Gamma_2$, then $P_{\Gamma_1} \subseteq P_{\Gamma_2}$;
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- Right reflexivity: if $\gamma \in \Gamma$, then $P_\Gamma(d, \gamma)$ for some $d \in D$;
- Monotonicity: if $\Gamma_1 \subseteq \Gamma_2$, then $P_{\Gamma_1} \subseteq P_{\Gamma_2}$;
- Compositionality: if $P_\Gamma(E, \Psi)$ and $P_\Psi(d, \varphi)$, then $P_\Gamma(E \circ d, \varphi)$.

Definition

A *sequent calculus given by derivations* is a pair $\mathcal{D} = \langle C, P \rangle$ where C is a signature and $P = \{P_\Delta : \Delta \in \wp_{\text{fin}}\mathsf{Seq}_C\}$ is a family of predicates $P_\Delta \subseteq \mathsf{Seq}_C^* \times \mathsf{Seq}_C$ such that the following conditions hold.

- Conclusion: if $P_\Delta(\omega, s)$ holds, then s is the first element in ω.
- Monotonicity: if $\Delta_1 \subseteq \Delta_2$, then $P_{\Delta_1} \subseteq P_{\Delta_2}$.
- Closure under substitution: if $P_\Delta(\omega, s)$ holds and σ is a substitution, then $P_{\sigma(\Delta)}(\sigma(\omega), \sigma(s))$ also holds.

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Induced calculus from rules

Let $\mathcal{R} = \langle C, R \rangle$ be a sequent calculus given by rules and define $\mathcal{D}(\mathcal{R}) = \langle C, P \rangle$ where $P_\Delta(\omega, s)$ holds iff ω is a rule-derivation of s from Δ.

Then $\mathcal{D}(\mathcal{R})$ is a sequent calculus given by derivations.

Furthermore, $\Delta \vdash_{\mathcal{R}} s$ iff $\Delta \vdash_{\mathcal{D}(\mathcal{R})} s$.

Translation

Definition

Let C and C' be signatures with $C \subseteq C'$ and $g : L(C') \to \mathbb{N}$ be an injection.

The *translation* $\tau_g : L(C') \to L(C)$ is a map defined inductively as follows:

- $\tau_g(\xi_i) = \xi_{2i+1}$ for $\xi_i \in \Xi$;
- $\tau_g(c(\gamma'_1, \ldots, \gamma'_k)) = c(\tau_g(\gamma'_1), \ldots, \tau_g(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$;
- $\tau_g(c'(\gamma'_1, \ldots, \gamma'_k)) = \xi_{2g(c'(\gamma'_1, \ldots, \gamma'_k))}$ for $c' \in C'_k \setminus C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$.

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With C, C' and g as above, $\tau_g^{-1} : \Xi \to L(C')$ is the following substitution:

- $\tau_g^{-1}(\xi_{2i+1}) = \xi_i$;
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The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C, P \rangle$, where $C = C' \cup C''$ and each P_Δ is inductively defined as follows.

- if $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds, then $P_\Delta(\omega, s)$ also holds;
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Example

We show that $\vdash_{\mathcal{D}(S4)\uplus\mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$

1.	$\longrightarrow (\Diamond''(\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$	Cut, 2, 5
2.	$(\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$	$\mathrm{R}\Diamond''$, 3
3.	$(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))) \longrightarrow (\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	R $\rightarrow$, 4
4.	$(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), \xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	Ax
5.	$\longrightarrow (\Diamond''(\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	RW, 6
6.	$\longrightarrow (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathrm{R}\Box''$, 7
7.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	Hyp
1.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	$\mathrm{R}\Diamond'$, 2
2.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1))$	R $\rightarrow$, 3
3.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1)$	$\mathrm{R}\Box'$, 4
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6.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1), \xi_1$	Ax

Theorem

Let $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ be sequent calculi given by rules such that Cut, LW and RW are in $R' \cup R''$, and define:

- $\mathcal{D}' = \mathcal{D}(\mathcal{R}')$ *and* $\mathcal{D}'' = \mathcal{D}(\mathcal{R}'')$ *are the sequent calculi given by derivations induced by* $\mathcal{R}'$ *and* $\mathcal{R}''$;
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Then $\mathcal{D}$ *and* $\mathcal{R}$ *are equivalent systems in the sense that* $\Delta \vdash_{\mathcal{R}} s$ *iff* $\Delta \vdash_{\mathcal{D}} s$, *for any* $\Delta \subseteq \mathrm{Seq}_C$ *and* $s \in \mathrm{Seq}_C$.

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Theorem (Characterization via rules)

A $\mathcal{R}$ be a sequent calculus given by rules is decidable iff for every rule r the relation S_r is recursive, where S_r is the relation such that $S_r(s_1, \ldots, s_n, s)$ holds iff $\langle \{s_1, \ldots, s_n\}, s \rangle$ is an instance of r.

Corollary

Let $\mathcal{R}'$ and $\mathcal{R}''$ be decidable sequent calculi given by rules.

Then their fibring $\mathcal{R} = \mathcal{R}' \uplus \mathcal{R}''$ is decidable.

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Theorem

Let $\mathcal{D}'$ and $\mathcal{D}''$ be decidable sequent calculi given by derivations.

Then their fibring $\mathcal{D} = \mathcal{D}' \uplus \mathcal{D}''$ is decidable.

Algorithm

- For each partition of ω do
  1. If the partition is singular, check whether $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds or $P''_{\tau''(\Delta)}(\tau''(\omega), \tau''(s))$ holds. If either is the case, output 1; otherwise move to the next partition.
  2. Otherwise, let ω^* be the first sequence in the partition and $\omega_1, \ldots, \omega_n$ the remaining ones. Let s_i denote $(\omega_i)_1$.
  3. For each $i = 1, \ldots, n$ check whether $P_{\Delta}(\omega_i, s_i)$ holds. If this is not the case, go on to the next partition.
  4. If the test above succeeded for all i, check whether $P_{(s_1,\ldots,s_n)}(\omega^*, s)$ holds. If this is the case, output 1.
- When no partitions of ω are left, output 0.

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- Preservation of cut-elimination
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Future work

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