The Essence of Proofs in Sequent Calculi

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Motivation

- use of logic to describe behaviour of systems
- different systems $\longleftrightarrow$ different logics
- combination of systems $\longleftrightarrow$

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Two simple examples (I)

The behaviour of system A is described by linear temporal logic with (state) propositional variables p and q.

The behaviour of system B is described by linear temporal logic with a (state) propositional variable r.

Under reasonable assumptions, the joint system can be described by linear temporal logic with state variables p, q and r.

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Two simple examples (II)

Epistemological logics (dealing with knowledge) typically include an $S5$ modality K.

Deontic logics (reasoning about obligation) use a D modality O.

Reasoning about Law requires the combination of these two logics, where one wants to write formulas mixing both modal operators.

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- more generally applicable;
- fewer restrictions on language.

Key results: preservation of properties

syntactical: decidability, complexity

semantical: finite model property, cardinality results, decidability

mixed: soundness, completeness

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An important distinction

Homogeneous fibring deals with combining two logics presented/defined in a similar way, e.g.:

- two Hilbert calculi;
- two sequent calculi;
- two logics given by semantics.

Heterogeneous fibring attempts to combine two logics presented/defined by different means, e.g.:

- a Hilbert calculus and a sequent calculus;
- a sequent calculus and a modal logic characterized by some class of Kripke structures.

Heterogeneous fibring is a much harder problem that has only recently been addressed.

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Remark: signatures

Throughout we will only consider logics with a propositional basis.

Definition

A propositional *signature* is a family $C = \{C_k\}_{k\in\mathbb{N}}$ of sets. Each $c_k \in C_k$ is called a *constructor* or *connective* of arity k.

The *language* $L(C)$ is the free algebra over C generated by a countable set $\Xi = \{\xi_n : n \in \mathbb{N}\}$ of meta-variables.

The elements of $L(C, \Xi)$ are called *formulas*.

We say that $C \subseteq C'$ if $C_k \subseteq C'_k$ for every $k \in \mathbb{N}$.

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A *sequent* is a pair $\Gamma \longrightarrow \Delta$, where Γ, Δ are multisets over $L(C)$

A *rule* is a pair $\frac{\theta_1,\ldots,\theta_n}{\gamma}$ where $\theta_1, \ldots, \theta_n, \gamma$ are sequents.

Definition

A *sequent calculus (given by rules)* is a pair $\mathcal{R} = \langle C, R \rangle$, where C is a signature and R is a set of rules including structural rules and specific rules (for the connectives).

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Structural rules

These are chosen *among* the following.

$$\frac{\xi_1, \Delta_1 \longrightarrow \Delta_2 \quad \Delta_1 \longrightarrow \Delta_2, \xi_1}{\Delta_1 \longrightarrow \Delta_2}\ \text{Cut}$$

$$\frac{\Delta_1 \longrightarrow \Delta_2}{\xi_1, \Delta_1 \longrightarrow \Delta_2}\ \text{LW} \qquad \frac{\Delta_1 \longrightarrow \Delta_2}{\Delta_1 \longrightarrow \Delta_2, \xi_1}\ \text{RW}$$

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Rules for the connectives

These *may* include:

- Left rules: the antecedent of the conclusion includes a formula $c(\varphi_1, \ldots, \varphi_n)$ for some n-ary connective c.
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Derivations

Definition

A *(rule-)derivation* of a sequent s from a set of sequents Θ in sequent calculus $\mathcal{R}$ is a finite sequence $\{\Gamma_i \longrightarrow \Delta_i\}_{i=1}^n$ of sequents such that:

- $\Gamma_1 \longrightarrow \Delta_1$ is s;
- for each $i = 1, \ldots, n$, one of the following holds:
  - $\Gamma_i \cap \Delta_i \neq \emptyset$ (justified by Ax);
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Example: $S4$

All structural rules plus:

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \to \xi_2), \Gamma \longrightarrow \Delta} \; \mathsf{L}\to \qquad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \to \xi_2)} \; \mathsf{R}\to$$

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$$\frac{\Box \Gamma_1 \longrightarrow \xi_1, \Delta_1}{\Gamma_2, \Box(\Gamma_1) \longrightarrow (\Box \xi_1), \Diamond(\Delta_1), \Delta_2}\ \mathrm{R}\Box \qquad \frac{\Gamma \longrightarrow \Delta, \xi_1, (\Diamond \xi_1)}{\Gamma \longrightarrow \Delta, (\Diamond \xi_1)}\ \mathrm{R}\Diamond$$

where $\Box(\Gamma) = \{(\Box \varphi) : \varphi \in \Gamma\}$ and $\Diamond(\Gamma) = \{(\Diamond \varphi) : \varphi \in \Gamma\}$

Derivation in $S4$

Example

The following shows that $\vdash_{S4} \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1)))$.

1.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1)))$	$R\Diamond, 2$
2.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\xi_1 \to (\Box\xi_1))$	$R \to, 3$
3.	$\xi_1 \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\Box\xi_1)$	$R\Box, 4$
4.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), \xi_1$	$R\Diamond, 5$
5.	$\longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\xi_1 \to (\Box\xi_1)), \xi_1$	$R \to, 6$
6.	$\xi_1 \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\Box\xi_1), \xi_1$	Ax

Example: D

All structural rules plus:

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \to \xi_2), \Gamma \longrightarrow \Delta}\ \mathsf{L}\to \qquad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \to \xi_2)}\ \mathsf{R}\to$$

$$\frac{\Gamma \longrightarrow \Delta, \xi_1}{\Gamma, (\neg\xi_1) \longrightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{\Gamma, \xi_1 \longrightarrow \Delta}{\Gamma \longrightarrow (\neg\xi_1), \Delta}\ \mathsf{R}\neg$$

$$\frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Box\xi_1)}\ \mathsf{R}\Box \qquad \frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Diamond\xi_1)}\ \mathsf{R}\Diamond$$

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Derivation in D

Example

The following shows that $\longrightarrow \xi_2 \vdash_D \longrightarrow (\Diamond(\xi_1 \to \xi_2))$

1.	$\longrightarrow (\Diamond(\xi_1 \to \xi_2))$	Cut, 2, 5
2.	$(\Box\xi_2) \longrightarrow (\Diamond(\xi_1 \to \xi_2))$	R$\Diamond$, 3
3.	$\xi_2 \longrightarrow (\xi_1 \to \xi_2)$	R $\to$, 4
4.	$\xi_2, \xi_1 \longrightarrow \xi_2$	Ax
5.	$\longrightarrow (\Diamond(\xi_1 \to \xi_2)), (\Box\xi_2)$	RW, 6
6.	$\longrightarrow (\Box\xi_2)$	R$\Box$, 7
7.	$\longrightarrow \xi_2$	Hyp

Definition

Let $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ be sequent calculi.

The (rule-)*fibring* $\mathcal{R}' \uplus \mathcal{R}''$ of $\mathcal{R}'$ and $\mathcal{R}''$ is the sequent calculus $\langle C' \cup C'', R' \cup R'' \rangle$.

Example

We can show that $\vdash_{S4 \uplus D} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$

1.	$\longrightarrow \Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))$	Cut, 2, 5
2.	$(\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$	$R\Diamond''$, 3
3.	$(\Diamond'(\xi_1 \to (\Box'\xi_1))) \longrightarrow (\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))$	$R \to$, 4
4.	$\xi_2, (\Diamond'(\xi_1 \to (\Box'\xi_1))) \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	Ax
5.	$\longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1))))$	RW, 6
6.	$\longrightarrow (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1))))$	$R\Box''$, 7
7.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	$R\Diamond'$, 8
8.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1))$	$R \to$, 9
9.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1)$	$R\Box'$, 10
10.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), \xi_1$	$R\Diamond'$, 11
11.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)), \xi_1$	$R \to$, 12
12.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1), \xi_1$	Ax

The problem

There is obviously a relation between the derivation above and the ones done in $S4$ and D... but how can we formalize that?

"Derivation" is a derived notion, whereas rules are primitive; but useful properties (cut elimination, decidability) are properties of derivations, not of rules...

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$\rightsquigarrow$ how about taking *derivations* as primitive objects?

Definition

A *sequent calculus given by derivations* is a pair $\mathcal{D} = \langle C, P \rangle$ where C is a signature and $P = \{P_\Theta : \Theta \in \wp_{\text{fin}}\text{Seq}_C\}$ is a family of predicates $P_\Theta \subseteq \text{Seq}_C^* \times \text{Seq}_C$ such that the following conditions hold.

- Conclusion: if $P_\Theta(\omega, s)$ holds, then s is the first element in ω.
- Monotonicity: if $\Theta_1 \subseteq \Theta_2$, then $P_{\Theta_1} \subseteq P_{\Theta_2}$.
- Closure under substitution: if $P_\Theta(\omega, s)$ holds and σ is a substitution, then $P_{\sigma(\Theta)}(\sigma(\omega), \sigma(s))$ also holds.

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Induced calculus from rules

Let $\mathcal{R} = \langle C, R\rangle$ be a sequent calculus given by rules and define $\mathcal{D}(\mathcal{R}) = \langle C, P\rangle$ where $P_\Theta(\omega, s)$ holds iff ω is a rule-derivation of s from Θ.

Then $\mathcal{D}(\mathcal{R})$ is a sequent calculus given by derivations.

Furthermore, $\Theta \vdash_{\mathcal{R}} s$ iff $\Theta \vdash_{\mathcal{D}(\mathcal{R})} s$.

Translation

Definition

Let C and C' be signatures with $C \subseteq C'$ and $g : L(C') \to \mathbb{N}$ be an injection.

The *translation* $\tau_g : L(C') \to L(C)$ is a map defined inductively as follows:

- $\tau_g(\xi_i) = \xi_{2i+1}$ for $\xi_i \in \Xi$;
- $\tau_g(c(\gamma'_1, \dots, \gamma'_k)) = c(\tau_g(\gamma'_1), \dots, \tau_g(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1, \dots, \gamma'_k \in L(C')$;
- $\tau_g(c'(\gamma'_1, \dots, \gamma'_k)) = \xi_{2g(c'(\gamma'_1, \dots, \gamma'_k))}$ for $c' \in C'_k \setminus C_k$ and $\gamma'_1, \dots, \gamma'_k \in L(C')$.

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Inverse translation

Definition

With C, C' and g as above, $\tau_g^{-1} : \Xi \to L(C')$ is the following substitution:

- $\tau_g^{-1}(\xi_{2i+1}) = \xi_i$;
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It is easy to check that $\tau^{-1} \circ \tau = \text{id}$ and $\tau \circ \tau^{-1} = \text{id}$.

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Let $\mathcal{D}' = \langle C', P' \rangle$ and $\mathcal{D}'' = \langle C'', P'' \rangle$ be sequent calculi given by derivations.

The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C, P \rangle$, where $C = C' \cup C''$ and each P_Θ is inductively defined as follows.

- if $P'_{\tau'(\Theta)}(\tau'(\omega), \tau'(s))$ holds, then $P_\Theta(\omega, s)$ also holds;
- if $P''_{\tau''(\Theta)}(\tau''(\omega), \tau''(s))$ holds, then $P_\Theta(\omega, s)$ also holds;
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Example

We show that $\vdash_{\mathcal{D}(S4)\uplus\mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$

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2.	$(\Box''\xi_2) \longrightarrow (\Diamond''(\xi_1 \to \xi_2))$	$R\Diamond''$, 3
3.	$\xi_2 \longrightarrow (\xi_1 \to \xi_2)$	$R \to$, 4
4.	$\xi_2, \xi_1 \longrightarrow \xi_2$	Ax
5.	$\longrightarrow (\Diamond''(\xi_1 \to \xi_2)), (\Box''\xi_2)$	RW, 6
6.	$\longrightarrow (\Box''\xi_2)$	$R\Box''$, 7
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1.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	$R\Diamond'$, 2
2.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1))$	$R \to$, 3
3.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1)$	$R\Box'$, 4
4.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), \xi_1$	$R\Diamond'$, 5
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5.	$\longrightarrow (\Diamond''(\xi_1 \to \xi_2)), (\Box''\xi_2)$	RW, 6
6.	$\longrightarrow (\Box''\xi_2)$	$\mathsf{R}\Box''$, 7
7.	$\longrightarrow \xi_2$	Hyp

1.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	$\mathsf{R}\Diamond'$, 2
2.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1))$	$\mathsf{R}\to$, 3
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We show that $\vdash_{\mathcal{D}(S4)\uplus\mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$

1.	$\longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$	Cut, 2, 5
2.	$(\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$	$\mathrm{R}\Diamond''$, 3
3.	$(\Diamond'(\xi_1 \to (\Box'\xi_1))) \longrightarrow (\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))$	$\mathrm{R} \to$, 4
4.	$(\Diamond'(\xi_1 \to (\Box'\xi_1))), \xi_1 \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	Ax
5.	$\longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1))))$	RW, 6
6.	$\longrightarrow (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1))))$	$\mathrm{R}\Box''$, 7
7.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	Hyp

1.	$\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	$\mathrm{R}\Diamond'$, 2
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Theorem

Let $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ be sequent calculi given by rules such that Cut, LW and RW are in $R' \cup R''$, and define:

- *$\mathcal{D}' = \mathcal{D}(\mathcal{R}')$ and $\mathcal{D}'' = \mathcal{D}(\mathcal{R}'')$ are the sequent calculi given by derivations induced by $\mathcal{R}'$ and $\mathcal{R}''$;*
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Then $\mathcal{D}$ and $\mathcal{R}$ are equivalent systems in the sense that $\Delta \vdash_{\mathcal{R}} s$ iff $\Delta \vdash_{\mathcal{D}} s$, for any $\Delta \subseteq \mathrm{Seq}_C$ and $s \in \mathrm{Seq}_C$.

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Definition

A sequent calculus given by rules $\mathcal{R} = \langle C, R \rangle$ has cut elimination iff, for any $\Delta \subseteq \mathsf{Seq}_C$ and $s \in \mathsf{Seq}_C$, whenever $\Delta \vdash_{\mathcal{R}} s$ there is a derivation ω for $\Delta \vdash_{\mathcal{R}} s$ that does not use the cut rule.

Theorem

Let $\mathcal{R}'$ and $\mathcal{R}''$ be sequent calculi given by rules with cut elimination.
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A sequent calculus given by derivations $\mathcal{D} = \langle C, P \rangle$ is *decidable* iff, for every recursive set $\Delta \subseteq \mathsf{Seq}_C$, the relation P_Δ is recursive.

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Theorem (Characterization via rules)

A $\mathcal{R}$ be a sequent calculus given by rules is decidable iff for every rule r the relation S_r is recursive, where S_r is the relation such that $S_r(s_1, \ldots, s_n, s)$ holds iff $\langle\{s_1, \ldots, s_n\}, s\rangle$ is an instance of r.

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Let $\mathcal{R}'$ and $\mathcal{R}''$ be decidable sequent calculi given by rules.

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Theorem

Let $\mathcal{D}'$ and $\mathcal{D}''$ be decidable sequent calculi given by derivations.

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Algorithm

- For each partition of ω do
  1. If the partition is singular, check whether $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds or $P''_{\tau''(\Delta)}(\tau''(\omega), \tau''(s))$ holds. If either is the case, output 1; otherwise move to the next partition.
  2. Otherwise, let ω^* be the first sequence in the partition and $\omega_1, \dots, \omega_n$ the remaining ones. Let s_i denote $(\omega_i)_1$.
  3. For each $i = 1, \dots, n$ check whether $P_\Delta(\omega_i, s_i)$ holds. If this is not the case, go on to the next partition.
  4. If the test above succeeded for all i, check whether $P_{[s_1,\dots,s_n]}(\omega^*, s)$ holds. If this is the case, output 1.
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- new definition of fibring for sequent calculi
- preservation of cut-elimination
- preservation of decidability

Future work

- generalization of the notion of sequent
- generalization beyond propositional signature

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