twenty-five comparators is optimal when sorting nine inputs
(and twenty-nine for ten)

luís cruz-filipe[1] michael codish[2]
michael frank[2] peter schneider-kamp[1]

[1]department of mathematics and computer science
university of southern denmark

[2]department of computer science
ben-gurion university of the negev, israel

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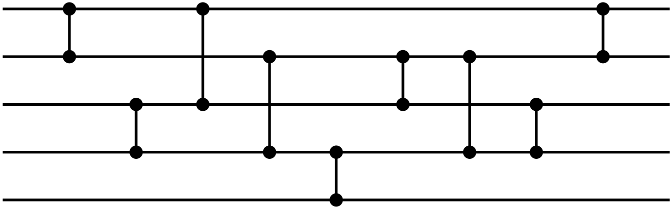
outline

a sorting network



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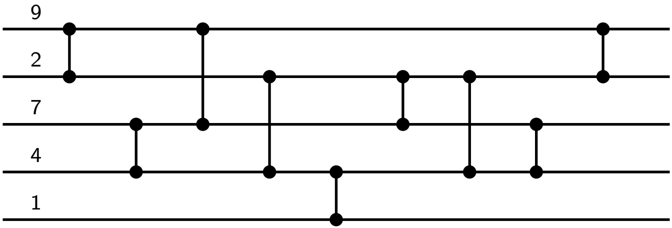

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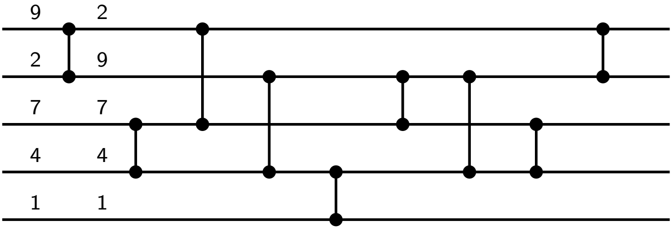

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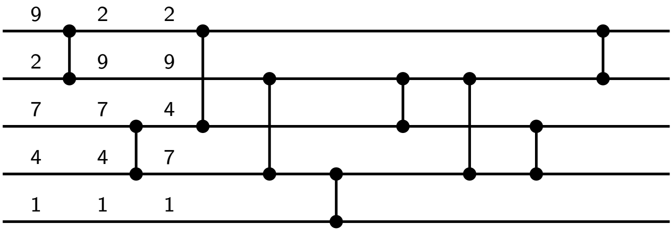

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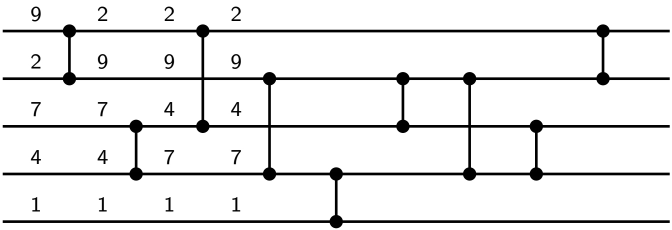

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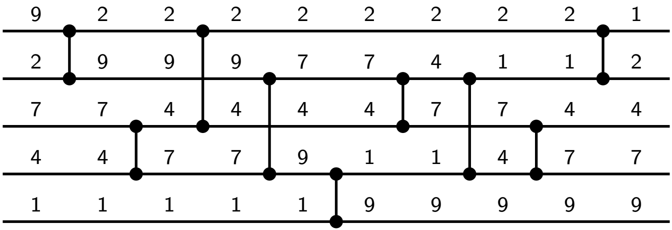

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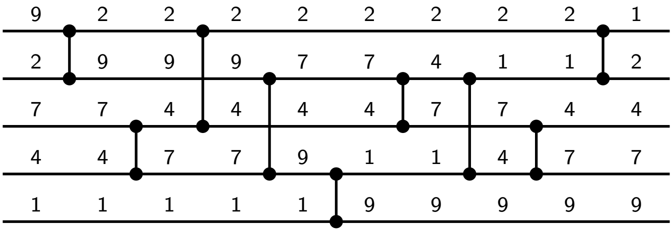

size this net has 5 *channels* and 9 *comparators*

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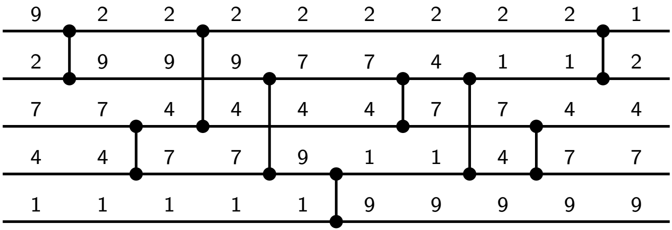

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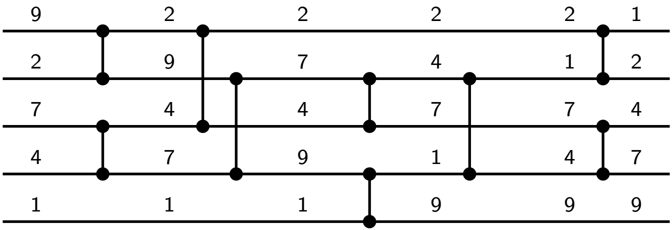

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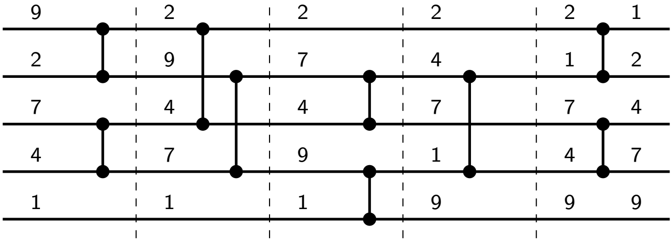

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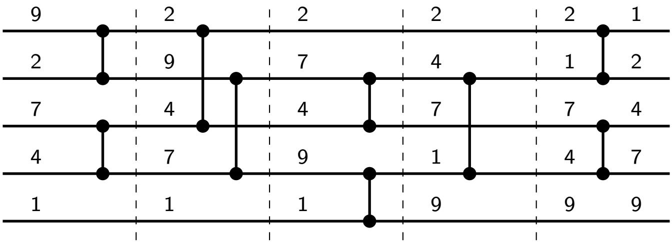

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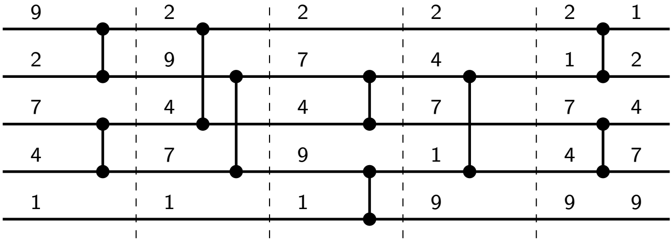

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more info see d.e. knuth, *the art of computer programming*, vol. 3

the optimization problems

the optimal size problem — what is the minimal number of *comparators* on a sorting network on n channels (s_n)?

the optimal depth problem — what is the minimal number of *layers* on a sorting network on n channels (t_n)?

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what is the minimal number of *layers* on a sorting network on n channels (t_n)?

knuth 1973

n	1	2	3	4	5	6	7	8	9	10
s_n	0	1	3	5	9	12	16	19	25 23	29 27
t_n	0	1	3	3	5	5	6	6	7 6	7 6

n	11	12	13	14	15	16
s_n	35 31	39 35	45 39	51 43	56 47	60 51
t_n	8 6	8 6	9 6	9 6	9 6	9 6

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parberry 1991

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bundala & závodný 2013

n	11	12	13	14	15	16
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t_n	**8**	**8**	**9**	**9**	**9**	**9**

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our contribution

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t_n	0	1	3	3	5	5	6	6	7	7

n	11	12	13	14	15	16
s_n	35 **33**	39 **37**	45 **41**	51 **45**	56 **49**	60 **53**
t_n	8	8	9	9	9	9

an exponential explosion

parberry 1991

- exploration of symmetries $\rightsquigarrow$ fixed first layer
- exhaustive search (200 hours of computation)

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bundala & závodný 2013

- reduced set of two-layer prefixes
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however... techniques not directly applicable to the size problem

9 channels (depth) 2620 possibilities for each layer
$\rightsquigarrow 2620^6 \approx 3.2 \times 10^{20}$ 6-layer networks

9 channels (size) 36 possibilities for each comparator
$\rightsquigarrow 36^{24} \approx 2.2 \times 10^{37}$ 24-comparator networks

outline

sorting networks in a nutshell

encoding the size problem in sat

conclusions & future work

comparator networks

comparator network — a *comparator network* C on n channels is a sequence of *comparators* (i, j) with $1 \leq i < j \leq n$

output — $C(\vec{x})$ denotes the *output* of C on $\vec{x} = x_1 \dots x_n$

binary outputs — the set of binary outputs of C is $\text{outputs}(C) = \{C(\vec{x}) \mid x \in \{0,1\}^n\}$

sorting network — a comparator network C is a *sorting network* if $C(\vec{x})$ is sorted for every input $\vec{x}$

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0–1 lemma (knuth 1973)	C is a sorting network on n channels iff C sorts all inputs in $\{0, 1\}^n$ "C is a sorting network on n channels" is co-NP (complete)

sat encoding

$$\texttt{Network} = \langle \texttt{c}(\texttt{I}_1, \texttt{J}_1), \ldots, \texttt{c}(\texttt{I}_k, \texttt{J}_k) \rangle$$

$$\texttt{valid}_{n,k}(\texttt{Network}) = \bigwedge_{i=1}^{k} \begin{array}{l} \texttt{new\_int}(\texttt{I}_i, 1, n) \\ \wedge\, \texttt{new\_int}(\texttt{J}_i, 1, n) \\ \wedge\, \texttt{int\_lt}(\texttt{I}_i, \texttt{J}_i) \end{array}$$

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$$\varphi_{\texttt{I},\texttt{J}}(\vec{x}, \vec{y}) = \bigwedge_{1 \leq i < j \leq n} \begin{array}{l} \texttt{int\_eq}(\texttt{I}, i) \land \texttt{int\_eq}(\texttt{J}, j) \\ \rightarrow ((y_i \leftrightarrow x_i \land x_j) \land (y_j \leftrightarrow x_i \lor x_j)) \end{array}$$

$$\psi_{\texttt{I},\texttt{J}}(\vec{x}, \vec{y}) = \bigwedge_{i=1}^{n} \begin{array}{l} \neg\texttt{int\_eq}(\texttt{I}, i) \land \neg\texttt{int\_eq}(\texttt{J}, i) \\ \rightarrow (y_i \leftrightarrow x_i) \end{array}$$

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$$\texttt{sorts}_{n,k}(\texttt{Network}, \vec{b}) = \bigwedge_{i=1}^{k} \begin{array}{l} \varphi_{\texttt{I}_i,\texttt{J}_i}(\vec{x}_{i-1}, x_i) \\ \land \psi_{\texttt{I}_i,\texttt{J}_i}(\vec{x}_{i-1}, \vec{x}_i) \end{array}$$

$$\vec{x}_0 = \vec{b}, \qquad \vec{x}_k = \mathsf{sort}(\vec{b})$$

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this is compiled with the bee constraint compiler into a cnf formula $\Psi(n, k)$

theorem $\Psi(n, k)$ is satisfiable iff there is a sorting network on n channels with k comparators

practical evaluation

	optimal sorting networks (sat)				
n	k	bee	#clauses	#vars	sat
4	5	0.18	1916	486	0.01
5	9	1.03	10159	2550	0.03
6	12	4.55	35035	8433	2.45
7	16	21.68	114579	26803	16.70
8	19	82.93	321445	73331	∞
9	25	452.55	977559	219950	∞

	smaller networks (unsat)				
n	k	bee	#clauses	#vars	sat
4	4	0.15	1480	356	0.01
5	8	0.90	8963	2221	1.27
6	11	3.99	32007	7657	242.02
7	15	19.04	107227	25000	∞
8	18	73.34	304145	69221	∞
9	24	406.67	937773	210715	∞

(times in seconds, timeout = 1 week)

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main idea divide the “big” sat problem into smaller problems

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reduce the size of $\mathcal{F}$ using symmetry-breaking techniques

breaking symmetry i/ii

output lemma (parberry 1991)

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permuted output lemma (bundala & Závodný 2013)

- C and C' comparator networks of depth 2
- $\pi(\text{outputs}(C)) \subseteq \text{outputs}(C')$ for some permutation π
- C' can be extended to a sorting network

then C can also be extended to a sorting network of depth 2

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permuted output lemma (generalized)

- C and C' comparator networks **of equal size**
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then C can also be extended to a sorting network of **the same size**

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subsumption $C \preceq C'$ when

$$\pi(\text{outputs}(C)) \subseteq \text{outputs}(C')$$

for some permutation π

the generate-and-prune approach

init set $R_0^n = \{\emptyset\}$ and $k = 0$

repeat until $k > 1$ and $|R_k^n| = 1$

- *generate* construct N_{k+1}^n by extending each net in R_k^n by one comparator in all possible ways
- *prune* construct R_{k+1}^n from N_{k+1}^n by keeping only one element of each minimal equivalence class w.r.t. the transitive closure of $\preceq$
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termination condition if C is a sorting network on n channels of size k, then $|R_k^n| = 1$

optimizations

- only generate networks when the extra comparator does something
- prove and implement criteria for when subsumption will fail
- restrict the search space of possible permutations
- optimize data structures
- parallelize to 288 nodes

some numerology

n	s_n	largest $\lvert N_k^n \rvert$	largest $\lvert R_k^n \rvert$	execution time
3	3	2	2	$\sim$0
4	5	12	4	$\sim$0
5	9	65	11	$\sim$0
6	12	380	53	2 sec
7	16	7,438	678	2 min
8	19	253,243	16,095	6 hours
9	25	18,420,674	914,444	16 years

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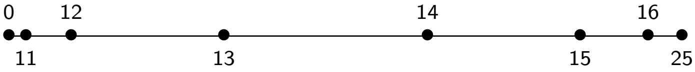
the hard part

going "over the peak" consumes most execution time



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taking $\mathcal{F} = R^9_{11}$ gives 188,730 *independent* problems that can be solved in parallel – which is much faster than letting the original program terminate

collaboration is the key

find "minimal" set $\mathcal{F}$ of choices for $\mathtt{I}_1$, $\mathtt{J}_1$,. . . , $\mathtt{I}_\ell$, $\mathtt{J}_\ell$ such that

$$\Psi_{n,k} \text{ is satisfiable iff } \bigvee_{f \in \mathcal{F}} \Psi_{n,k,f} \text{ is satisfiable}$$

where $f = \langle \mathtt{I}_1, \mathtt{J}_1, \ldots, \mathtt{I}_\ell, \mathtt{J}_\ell \rangle$

taking $\mathcal{F} = R^9_{11}$ gives 188,730 *independent* problems that can be solved in parallel – which is much faster than letting the original program terminate

different values of k give different total running times

outline

results & future work

- exact values of s_9 and s_{10}
- technique may be adapted to settle higher values which are still unknown
- algorithms may be useful for *finding* smaller-than-currently-known networks
- further theoretical results may help proving optimality of best known upper bounds

thank you!