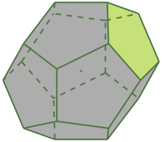
Mathematical Optimization



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MatØk Studiepratik

Operations Research

Operation Research (aka, Management Science, Analytics):

- the discipline that uses a **scientific approach to decision making**.
- It seeks to determine how best to design and operate a system, usually under conditions requiring the allocation of scarce resources, by means of **mathematics** and **computer science**.
- **Quantitative methods for planning and analysis.**

Applications:

- Transport
- Supply Chains
- Sport
- Finance
- Government
- Manufacturing

Today's Objectives

- Convert a problem from ordinary language into mathematical language
- Solve geometrically a system of linear inequalities and connect the solutions to the real world problem
- Distinguish Linear vs Non-linear, Continuous vs Integer problems
- Solve numerically the problems in Google Sheets and Microsoft Excel
- Applications: production planning, diet planning, budget allocation

$\rightsquigarrow$ Transmit to you my fascination for Mathematical Optimization

Outline

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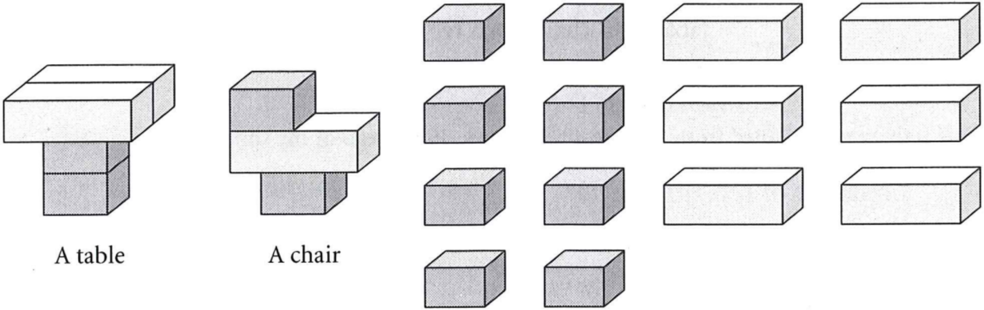
1. Production Planning
2. Diet Problem
3. Budget Allocation
4. Summary

Production Planning

Suppose a company produces only tables and chairs.
A table is made of 2 large Lego pieces and 2 small pieces, while a chair is made of 1 large and 2 small pieces.
The resources available are 8 small and 6 large pieces.



The profit for a table is 1600 dkk and for a chair 1000 dkk. What product mix maximizes the company's profile using the available resources?

Mathematical Model

	Tables	Chairs	Capacity
Small Pieces	2	2	8
Large Pieces	2	1	6
Profit	16	10	

Decision Variables

$x_1 \geq 0$ units of small pieces
$x_2 \geq 0$ units of large pieces

Object Function

$\max\ 16x_1 + 10x_2$ maximize profit

Constraints

$$\begin{array}{rcrcl} 2x_1 & + & 2x_2 & \leq & 8 \\ 2x_1 & + & x_2 & \leq & 6 \end{array}$$

small pieces capacity
large pieces capacity

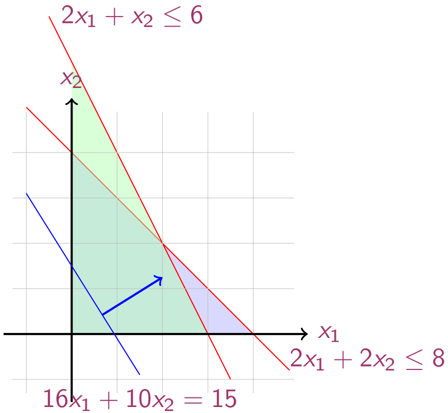
Mathematical Model

Materials A and B
Products 1 and 2

$$\begin{array}{lrcrcl} \max & 16x_1 & + & 10x_2 & & \\ & 2x_1 & + & 2x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 6 \\ & & & x_1 & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

a_{ij}	1	2	b_i
A	2	2	8
B	2	1	6
c_j	16	10	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$1, 2, \ldots, n$	products
$1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
$c_j = \sigma_j - \sum_{i=1}^{n} \rho_i a_{ij}$	profit per unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
x_j	amount of product j to produce

$$
\begin{array}{rlllllllll}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + \ldots + & c_nx_n & = & z \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + \ldots + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + \ldots + & a_{2n}x_n & \leq & b_2 \\
 & \ldots & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + \ldots + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

In Matrix Form

$$
\begin{array}{rcccccccccccc}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$
\begin{array}{rrcl}
\max & z & = & c^T x \\
 & Ax & = & b \\
 & x & \geq & 0
\end{array}
$$

Vector and Matrices in Excel

$$\sum_{j=1}^{n} c_j = c_1 + c_2 + \ldots + c_n$$

$\mathtt{SUM(B5:B14)}$

Scalar product

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \\ &= \sum_{j=1}^{n} u_j v_j \end{aligned}$$

$\mathtt{SUMPRODUCT(B5:B14, C5:C:14)}$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m \\ & x_j \ge 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 16x_1 + 10x_2 \\ & 2x_1 + 2x_2 \le 8 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 16 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \ge 0$$

Outline

1. Production Planning

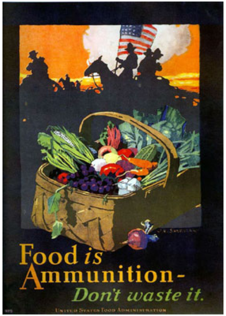
2. Diet Problem

3. Budget Allocation

4. Summary

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem
- (programming intended as planning not computer code)



min   cost/weight
subject to nutrition requirements:
    eat enough but not too much of Vitamin A
    eat enough but not too much of Sodium
    eat enough but not too much of Calories
    ...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread,
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50,000)

Food	Corn	2% Milk	Wheat bread
Vitamin A	107	500	0
Calories	72	121	65
Cost per serving	\$0.18	\$0.23	\$0.05

The Mathematical Model

Parameters (given data)

F = set of foods
N = set of nutrients

a_{ij} = amount of nutrient j in food i, $\forall i \in F$, $\forall j \in N$
c_i = cost per serving of food $i, \forall i \in F$
F_{mini} = minimum number of required servings of food $i, \forall i \in F$
F_{maxi} = maximum allowable number of servings of food $i, \forall i \in F$
N_{minj} = minimum required level of nutrient $j, \forall j \in N$
N_{maxj} = maximum allowable level of nutrient $j, \forall j \in N$

Decision Variables

x_i = number of servings of food i to purchase/consume, $\forall i \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{i \in F} c_i x_i$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \qquad \forall j \in N$$

Constraint Set 2: For each nutrient $j \in N$, do not exceed the maximum allowable level.

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \qquad \forall j \in N$$

Constraint Set 3: For each food $i \in F$, select at least the minimum required number of servings

$$x_i \geq F_{mini}, \qquad \forall i \in F$$

Constraint Set 4: For each food $i \in F$, do not exceed the maximum allowable number of servings.

$$x_i \leq F_{maxi}, \qquad \forall i \in F$$

The Mathematical Model

system of equalities and inequalities

$$\min \sum_{i \in F} c_i x_i$$

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \qquad \forall j \in N$$

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \qquad \forall j \in N$$

$$x_i \geq F_{mini}, \qquad \forall i \in F$$

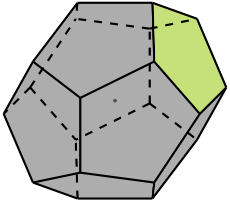
$$x_i \leq F_{maxi}, \qquad \forall i \in F$$

The History of Stigler's Diet Problem

- The linear program consisted of 9 equations in 77 variables
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution

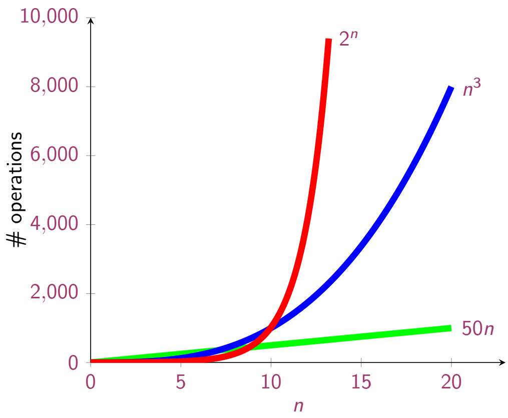
Geometrical Interpretation

Geometrically the feasibility region of a linear programming problem with 3 variables is a polyhedron.



The generalization of a polyhedron in n dimensions is called polytope.

Growth Functions



NP-hard problems: bad if we have to solve them, good for cryptology

Outline

Budget Allocation

- A company has six different opportunities to invest money.
- Each opportunity requires a certain investment over a period of 6 years or less.

Expected Investment Cash Flows and Net Present Value									
	Opp. 1	**Opp. 2**	**Opp. 3**	**Opp. 4**	**Opp. 5**	**Opp. 6**			**Budget**
Year 1	-$5.00	-$9.00	-$12.00	-$7.00	-$20.00	-$18.00			$45.00
Year 2	-$6.00	-$6.00	-$10.00	-$5.00	$6.00	-$15.00			$30.00
Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00			$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00			$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00			$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00			$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93			

- The company wants to invest in those opportunities that maximize the combined Net Present Value (NPV).
- It also has an investment budget that needs to be met for each year.

Net Present Value

- P: value of the original payment presently due
- the debtor wants to delay the payment for t years,
- let r be the market rate of return on a similar investment asset
- the future value of P is $F = P(1+r)^t$

Viceversa, consider the task of finding:

- the present value P of \$100 that will be received in five years, or equivalently,
- which amount of money today will grow to \$100 in five years when subject to a constant discount rate.

Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1+r)^t} = \frac{\$100}{(1+0.05)^5} = \$78.35.$$

Budget Allocation

Net Present Value calculation:
for each opportunity we calculate the NPV at time zero (the time of decision) as:

$$P_0 = \sum_{t=1}^{5} \frac{F_t}{(1+0.05)^5}$$

Expected Investment Cash Flows and Net Present Value									
	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6			Budget
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Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00			$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00			$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00			$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00			$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93			

Budget Allocation - Mathematical Model

- Let B_t be the budget available for investments during the years $t = 1..5$.
- Let a_{tj} be the cash flow for opportunity j and c_j its NPV
- Task: choose a set of opportunities such that the budget is never exceeded and the expected return is maximized. Consider both the case of indivisible and divisible opportunities.

Variables $x_j = 1$ if opportunity j is selected and $x_j = 0$ otherwise, $j = 1..6$

Objective

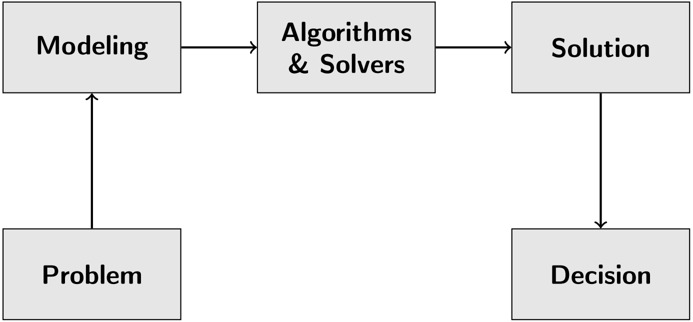
$$\max \sum_{j=1}^{6} c_j x_j$$

Constraints

$$\sum_{j=1}^{6} a_{tj} x_j + B_t \geq 0 \qquad \forall t = 1..5$$

Outline

The OR Journey



Mathematical Modeling

- Find out exactly what the decision makers need to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Decision Variables of suitable type (continuous, integer valued, binary) corresponding to the needs
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

Recognize linear and non linear functions and continuous and integer variables.

Solution Process

- Geometrical interpretation of the simplex method
- Touched computational issues
- Computer carries out the operations, hence programming needed
- Practical experience with Spreadsheets

Summary

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