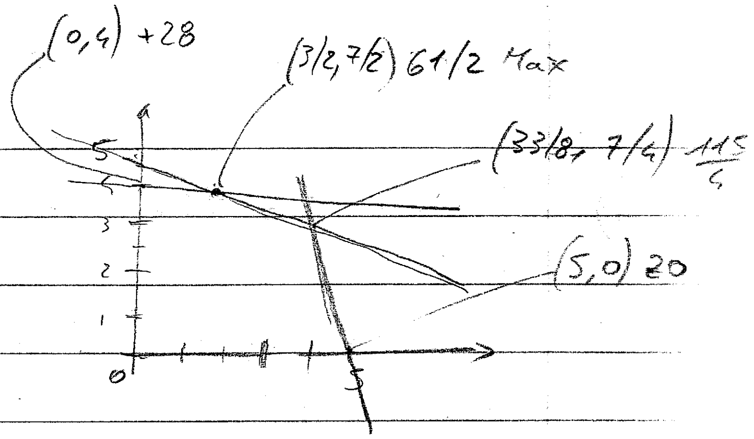
TASK 1



$\max z = 4x_1 + 7x_2$

$x_1 + 3x_2 \le 12$

$-4x_1 + 6x_2 \le 27$

$4x_1 + 2x_2 \le 20$

(a) $(0,0)$ is feasible $\Rightarrow z = 0 = LB$

(c)
(d)

x_1	x_2	s_1	s_2	s_3	$-z$	b	b_i/a_{i1}	b_i/a_{i2}	$c_j \cdot b_i/a_{ij}$
1	3	1	0	0	0	12	12	4	$7\cdot4=28$
4	6	0	1	0	0	27	$27/4 \simeq 8$	$\frac{27}{6} \simeq 4$	
4	2	0	0	1	0	20	5	10	$5\cdot4=20$
4	7	0	0	0	1	0			

largest coefficient $\rightsquigarrow 7$

largest increase $\rightsquigarrow 7$

1/3	1	1/3	0	0	0	4
2	0	-2	1	0	0	3
10/3	0	-2/3	0	1	0	12
-5/3	0	7/3	0	0	1	28

(e)

0	1	2/3	-1/6	0	0	7/2
1	0	-1	1/2	0	0	3/2
0	0	8/3	-5/3	1	0	7
0	0	-2/3	-5/6	0	1	-61/2

$GAP = \frac{61/2 - 0}{61/2} = 1$

(f) I row: $\frac{2}{3}s_1 + \frac{5}{6}s_2 \geq \frac{1}{2}$ $\qquad \frac{1}{2}s_1 \geq \frac{1}{2}$

$$\frac{2}{3}(12 - x_1 - 3x_2) + \frac{5}{6}(27 - 4x_1 - 6x_2) \geq \frac{1}{2}$$

$$8 - \frac{2}{3}x_1 - 2x_2 + \frac{135}{6} - \frac{10}{3}x_1 - 5x_2 \geq \frac{1}{2}$$

$$-4x_1 - 7x_2 \geq \frac{1}{2} - 8 - \frac{135}{6}$$

II row: $\frac{1}{2}s_2 \geq \frac{1}{2}$ $\qquad$ (i)-form

$$\frac{1}{2}(27 - 4x_1 - 6x_2) \geq 1$$

$$-4x_1 - 6x_2 \geq -26$$

$$-2x_1 - 3x_2 \geq -13$$

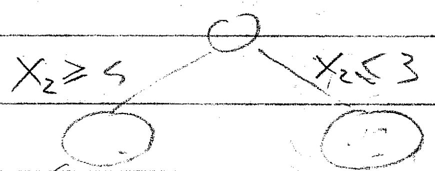
$$2x_1 + 3x_2 \leq 13$$ $\qquad$ (ii)-form

(g) Introducing the i-form we have a basis but infeasible.
Introducing the ii-form we do elementary row operations to arrive to a canonical form with infeasible basis.
We use dual simplex:

$$\begin{array}{cccccccc}
0 & 0 & 0 & \boxed{-1/2} & 0 & 0 & 1 & -1/2 \\
0 & 1 & 2/3 & -1/6 & 0 & 0 & 0 & 7/2 \\
1 & 0 & -1 & 1/2 & 0 & 0 & 0 & 3/2 \\
0 & 0 & 8/3 & -5/3 & 1 & 0 & 0 & 7 \\
0 & 0 & -2/3 & -5/6 & 0 & 1 & 0 & -61/2
\end{array}$$

1) pivot < 0
2) row with b term negative
3) col that $\min \left| \frac{c_j}{a_{ij}} \right|$ $\Rightarrow$ $-\frac{1}{2}$ is pivot

(h) Using branch and bound on we have



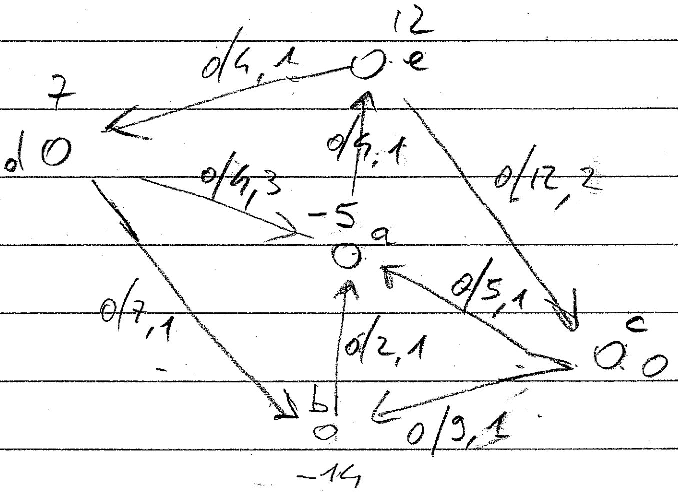
we could use heuristics again at each node

the LP requires a dual-simplex step
sol. will be (0,4) of val 28

sol will be (2,3) of val $\underline{\underline{29}}$

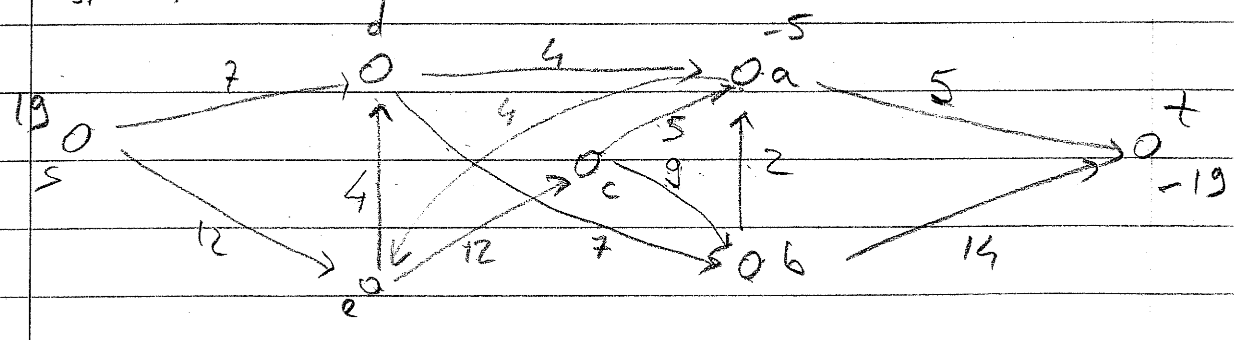
TASK 2

(a)

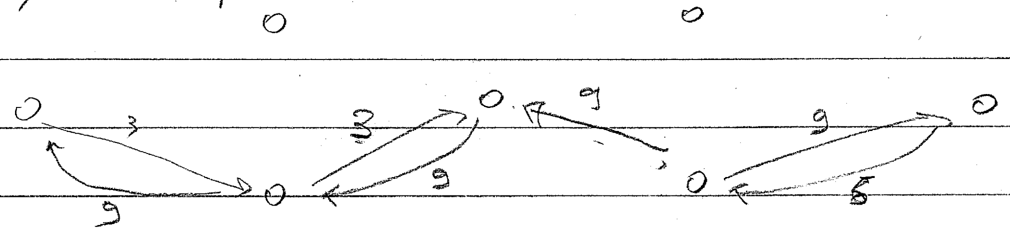


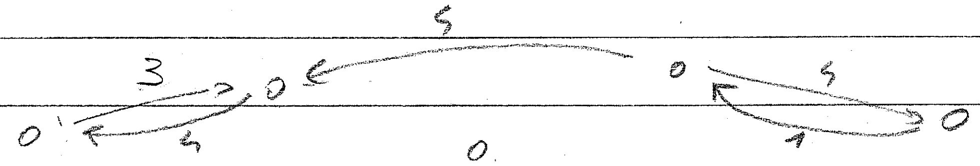
$l = 0$

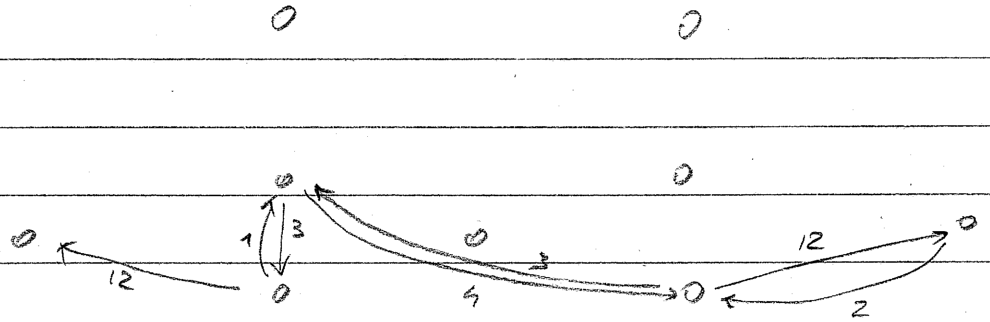
N_{st} network:

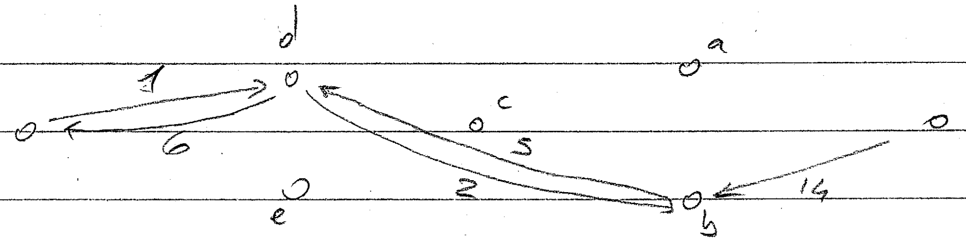


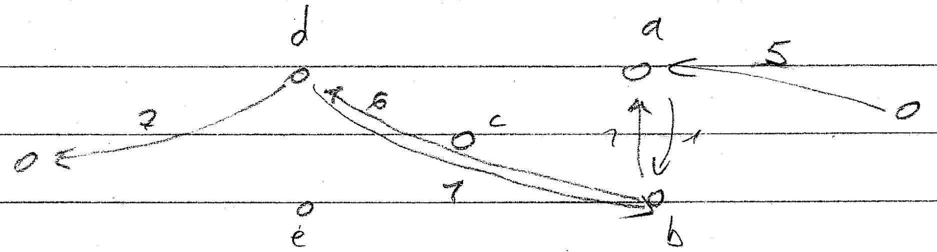
Apply max flow to Nst network. Ford Fulkerson alg:



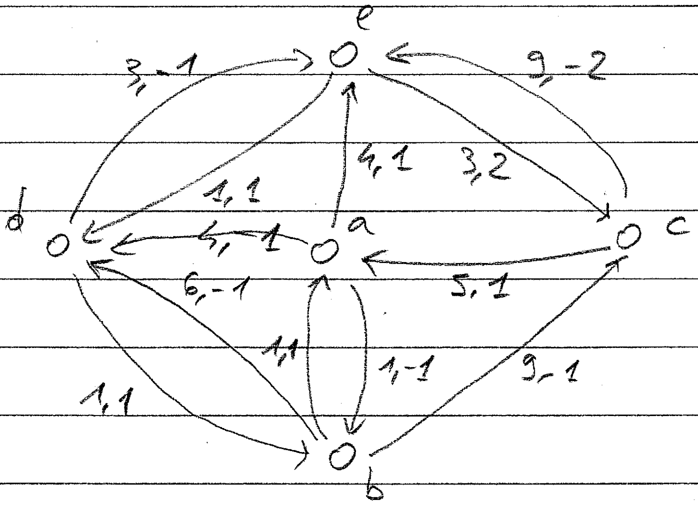






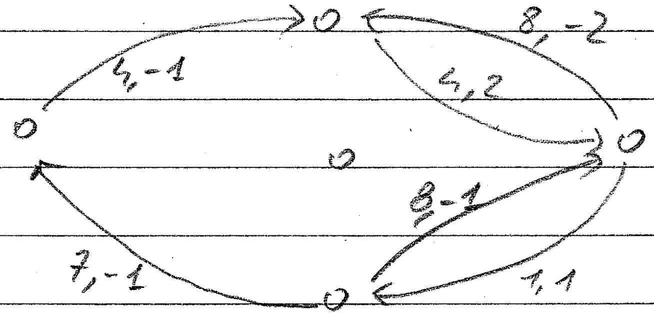


(b) $N(x)$

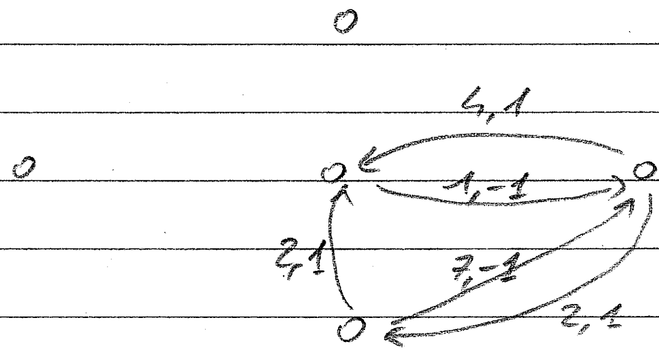


cost 41

(c)

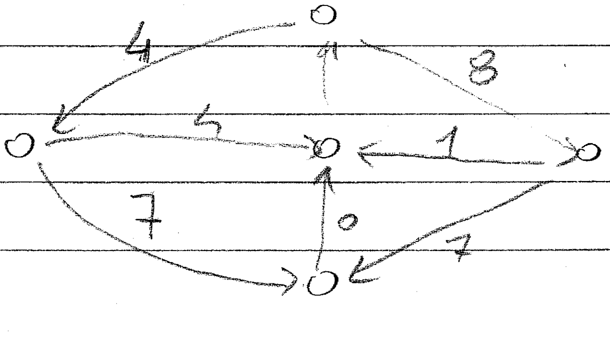


cost 40



cost 39

final flow



TASK 3

a

$$\begin{array}{c|c|c} A_B & A_N & b \\ \hline c_B & c_N & \end{array}$$

$$A_B x_B + A_N x_N = b$$

$$x_B = A_B^{-1} b$$

$$\bar{c}_B^T = c_B^T - \pi A_B = 0 \Rightarrow c_B^T A_B^{-1} = \pi$$

$$\bar{c}_N^T = c_N^T - \pi A_N = c_N^T - c_B^T A_B^{-1} A_N$$

$$\begin{array}{ccccccccc} & x_2 & x_3 & & & & S_3 & & \\ \end{array}$$
$$\begin{bmatrix} 2 & 1 & 5 & 1 & 1 & 0 & 0 & 0 & 8 \\ 2 & 2 & 0 & 4 & 0 & 1 & 0 & 0 & 12 \\ 3 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 18 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_B = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad A_B^{-1} = \begin{bmatrix} \frac{2}{10} & 0 & -\frac{4}{10} \\ -\frac{1}{10} & \frac{5}{10} & -\frac{3}{10} \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1,6 - 1,2 \\ 6 \\ -3,2 - 3,6 + 18 \end{bmatrix} = \begin{bmatrix} 0,4 \\ 6 \\ 11,2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ S_1 \end{bmatrix}$$

$$\bar{c}_N^T = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \text{"} - \text{"} \begin{bmatrix} \cancel{0,4 - 0,2} & \cancel{0,2 - 0,4} & \cancel{0,2} & \cancel{-0,1} \\ 1 & 2 & 0 & 0,5 \\ -0,8 - 0,6 + 3 & -0,4 - 1,2 & -0,4 & -0,3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0,2 & -0,2 & 0,2 & -0,1 \\ 1 & 2 & 0 & 0,5 \\ 1,6 & -1,6 & -0,4 & -0,3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0,2+2 & -0,2+4 & 0,2 & -0,1+1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1,2 & -2,8 & -0,2 & -0,9 \end{bmatrix}$$

All red. costs are negative $\Rightarrow$ sol. is optimal.

b) $$\pi^T = c_B^T A_B^{-1} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} = \begin{bmatrix} 0,2 & 0,9 & 0 \end{bmatrix}$$

The dual variables are the reduced costs of slack variables with change of sign. (This has been shown in the proof of the Strong duality Theorem).

$$y^T = \begin{bmatrix} 0,2 & 0,9 & 0 \end{bmatrix} = \pi^T$$

c) Looking at the dual var, the first and second constr. are binding. The third has a slack:

$$b_i - \sum_{j=1}^{n} a_{ij} x_j > 0$$

d) $$z^T = y^T b = \begin{bmatrix} 0,2 & 0,9 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix} = 1,6 + \quad + 0 = 23,4$$

new obj

before it was

$$c^T x^* = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 0,4 \\ 0 \\ 0 \\ 11,2 \end{bmatrix} = 12 + 0,4 = 12,4$$

hence obj increases

But we must make sure the b term does not become negative.

$$\bar{b} = A_B^{-1} b = \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix} = \begin{bmatrix} 1,6 - 2,6 \\ 13 \\ -3,2 - 7,8 + 18 \end{bmatrix} = \begin{bmatrix} -1 \\ 13 \\ 7 \end{bmatrix}$$

one term becomes negative ⇒ we need to iterate.

ⓔ we look back at how we computed red. costs.

$$\bar{c}_N = [\,3 \quad 1 \quad 0 \quad 0\,] - [\,2,2 \quad 3,8 \quad 0,2 \quad 0,9\,] =$$
$$= [\,0,8 \; \ldots \,]$$

the red. cost of x_1 becomes positive ⇒ x_1 enters the basis.

To determine the leaving var we need to see how much we can increase:

$$X_B = X_B^* - A_B^{-1} A_N X_N = X_B^* - A_B^{-1} \underset{\text{entering col.}}{a_e} X_N$$

$$A_B^{-1} \cdot a = \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0,4 - 0,2 \\ 1 \\ -0,8 - 0,6 + 3 \end{bmatrix} = \begin{bmatrix} 0,2 \\ 1 \\ 1,6 \end{bmatrix}$$

$$X_B = \begin{bmatrix} 0,4 \\ 6 \\ 11,2 \end{bmatrix} - \begin{bmatrix} 0,2 \\ 1 \\ 1,6 \end{bmatrix} t \geq 0$$

$t \leq 0,4/0,2 = 0,2 \leftarrow$

$t \leq 6$

$t \leq 11,2/1,6 = 7$

x_3 leaves the basis

(8) We need to determine the red. cost for the new var.

$$\bar{c}_N^T = c_N^T - \pi A_N =$$

$$= \begin{bmatrix} \delta & 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0{,}2 & 0{,}9 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 & 1 & 0 \\ 5 & 2 & 4 & 0 & 1 \\ 6 & 3 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \delta - 0{,}6 - 4{,}5 > 0$$

$$\delta > 5{,}1$$

TASK 4

(a)

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & y \end{array} \\ \left[\begin{array}{cccccc} & & & & & 1 \\ & & 1 & & & -1 \\ & 1 & & & & -1 \\ 1 & & & & & -1 \\ & & & 1 & & -1 \\ & & & & 1 & -1 \end{array}\right] \end{array} \begin{array}{l} \\ \leq 1 \\ \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \end{array}$$

$$A^T \text{ is } \left[\begin{array}{ccccc} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & -1 & -1 & -1 & -1 \end{array}\right] \Rightarrow \text{TUM for } I_1 = \text{all ro..}, \; I_2 = \emptyset$$

$$\sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij} = 0 \quad \forall j$$

the first column has only one entry not zero. However since the matrix without that column is TUM, all its square submatrices have determinant 0,1,-1, hence also all matrices obtained by adding the first column will have determinants 0,1,-1 since all its elements are 0 or 1.

b)

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 20 \\ 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 30 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for each $i = 4, 3 .. 1$ subtrack ith row with $(i+1)$th row

$$\begin{matrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 10 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 10 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -15 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -15 \end{matrix}$$

$\Rightarrow$ TUM and min cost flow

c)

$$\min \quad 720x_1 + 800x_2 + 760x_3 + 680x_4 + 720x_5 + 780x_6 + 640x_7$$

0 - 6 :	$x_1 + x_2$	≥ 2
6 - 8 :	$x_2 + x_3$	≥ 8
8 - 11 :	$x_3 + x_4$	≥ 5
11 - 14 :	$x_3 + x_4$	≥ 7
14 - 16 :	$x_4 + x_5$	≥ 3
16 - 18 :	$x_5 + x_6$	≥ 4
18 - 20 :	$x_5 + x_6 + x_7$	≥ 6
20 - 22 :	$x_5 + x_6 + x_7$	≥ 3
22 - 24 :	$x_6 + x_7$	≥ 1

$x_1, x_2 \dots x_7 \geq 0$ and integer.

The matrix has consecutive 1's property on cols hence LP relax gives integer results.

this solution contains the mistakes indicated in red.
See next page for correct solution.

TASK 5

(a)

$$(P)\quad \begin{aligned} \max\ & x_1 - 2x_2 \\ & x_1 + 2x_2 - x_3 \geq 0 \\ & 4x_1 + 3x_2 + 4x_3 \leq 3 \\ & 2x_1 - x_2 + 2x_3 = 1 \\ & x_2, x_3 \geq 0 \end{aligned}$$

$$P(y_1,y_2,y_3) = \max\ x_1 - 2x_2 + y_1(x_1 + 2x_2 - x_3) + y_2(4x_1 + 3x_2 + 4x_3 - 3) + y_3(2x_1 - x_2 + 2x_3 - 1)$$

$$\forall\ y_1 \geq 0,\ y_2 \leq 0,\ y_3 \in \mathbb{R} \qquad P(y_1, y_2, y_3) \geq opt(P)$$

$$\min_{y_1, y_2, y_3 \in \mathbb{R}}\ P(y_1, y_2, y_3)$$

$$\begin{aligned} \max\ & (1 + y_1 + 4y_2 + 2y_3)\, x_1 \\ & (-2 + 2y_1 + 3y_2 - y_3)\, x_2 \\ & (-y_1 - 4y_2 + 2y_3)\, x_3 \\ & -3y_2 - y_3 \end{aligned}$$

+

$\max\ c_1x_1 + c_2x_2 + \ldots \Rightarrow$ if $c_i > 0 \Rightarrow x_i = \infty$

$c_i < 0 \Rightarrow x_i = 0$

Hence:

if $x \in \mathbb{R} \Rightarrow c_i = 0$ or else unbounded

$$\begin{aligned} \min\ & -3y_2 - y_3 \\ & y_1 + 4y_2 + 2y_3 \leq -1 \\ & 2y_1 + 3y_2 - y_3 \leq 2 \\ & -y_1 - 4y_2 + 2y_3 = 0 \\ & y_1 \geq 0 \quad y_2 \leq 0 \quad y_3 \gtrless 0 \end{aligned}$$

=

<=

$\boxed{\text{TASK 5}}$

(a)

$$\max \quad x_1 - 2x_2$$
$$(P) \quad \begin{aligned} & x_1 + 2x_2 - x_3 \geq 0 \\ & 4x_1 + 3x_2 + 4x_3 \leq 3 \\ & 2x_1 - x_2 + 2x_3 = 1 \\ & x_2, x_3 \geq 0 \\ & x_1 \in \mathbb{R} \end{aligned}$$

We want an UB; we bring (relax) constraints in obj. func.

$$P(y_1, y_2, y_3) = \max\ x_1 - 2x_2 + y_1(x_1 + 2x_2 - x_3) + y_2(4x_1 + 3x_2 + 4x_3 - 3) + y_3(2x_1 - x_2 + 2x_3 - 1)$$

$\forall\, y_1 \geq 0,\ y_2 \leq 0,\ y_3 \in \mathbb{R} \qquad P(y_1, y_2, y_3) \geq opt(P)$

(this proves the weak duality th. by construction)

$\min\limits_{y_1 y_2 y_3} P(y_1\, y_2\, y_3)$ this will give us the best UB.

$$\max\ (1 + y_1 + 4y_2 + 2y_3)\, x_1 + (-2 + 2y_1 + 3y_2 - y_3)\, x_2 + (-y_1 + 4y_2 + 2y_3)\, x_3 - 3y_2 - y_3$$

This problem can be solved by inspection.
It is of the form:

$\max\ c_1 x_1 + c_2 x_2 + \dots + c_n x_n \Rightarrow$
- if $c_i > 0 \wedge x_i \geq 0 \Rightarrow x_i = +\infty$
- if $c_i \leq 0 \wedge x_i \geq 0 \Rightarrow x_i = 0$, bounded
- if $c_i \neq 0 \wedge x_i \in \mathbb{R} \Rightarrow$ unbounded
- if $c_i = 0 \wedge x_i \in \mathbb{R} \Rightarrow$ bounded

We are only interested in the cases where the problem is bounded hence:

$$\begin{aligned} \min\ & -3y_2 - y_3 \\ & y_1 + 4y_2 + 2y_3 = -1 \\ & 2y_1 + 3y_2 - y_3 \leq 2 \\ & -y_1 + 4y_2 + 2y_3 \leq 0 \\ & y_1 \geq 0 \\ & y_2 \leq 0 \\ & y_3 \in \mathbb{R} \end{aligned} \quad \equiv \quad \begin{aligned} \min\ & +3y_2 + y_3 \\ & -y_1 + 4y_2 - 2y_3 = -1 \\ & -2y_1 - 3y_2 + y_3 \leq 2 \\ & +y_1 - 4y_2 + 2y_3 \leq 0 \\ & y_1 \leq 0 \\ & y_2 \geq 0 \\ & y_3 \in \mathbb{R} \end{aligned}$$

TASK 6

a) it is a transportation problem:

$$\min \sum_{ij} c_{ij} x_{ij}$$

for

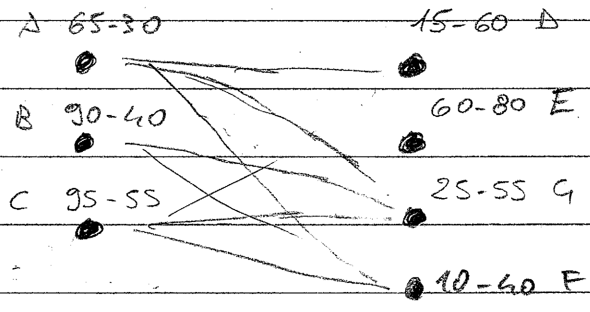
$$\sum_j x_{ij} - \sum_j x_{ji} = f_i - d_i \quad \forall i$$

desired

current

x_{ij} integer

This is a network flow problem, specifically a Transportation problem hence any alg. for it would go:



those who have an ~~excess~~ of cars

those who require cars to be added