LOCAL SEARCH METHODS
APPLICATIONS AND ENGINEERING

Lecture 12

Empirical Methods
Statistical Inference

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Outline

1. Statistical Inference
2. Two Sample Tests
3. Design of Experiments for Algorithms
4. Analysis of Variance

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Where do we need inference?

- in the prediction of algorithm results
- in the comparison between algorithms
- in the analysis of the impact of algorithmic factors
- in the goodness of fit of distributions

Statistical Inference

Random Sample	Inference	Population
X^n	$\longrightarrow$	$P(x, \theta)$
Statistical Estimator $\widehat{\theta}$		Parameter θ

Deductive approach: test hypotheses on the parameters of the population

Inductive approach: estimate unknown parameters of the population (through confidence intervals)

Parameter Estimation

Estimator $\hat{\theta}(X_1, \ldots, X_n)$ makes a guess on the parameter (Es. $\bar{X}$)

Estimate is the actual value $\hat{\theta}(x_1, \ldots, x_n)$

Properties of an estimator:

- unbiased: $E[\hat{\theta}] = \theta$ (*e.g.*, $E[\bar{X}] = \mu$)
- consistent
- efficient (uncertainty must decrease with size, *e.g.*, Var$[\bar{X}] = \sigma^2/n$)
- sufficient

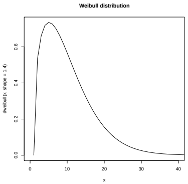
Note: The *best* result $b_N = \min_i c_i$ is not a good estimator. It is biased and not efficient.

Theorem: Central Limit Theorem

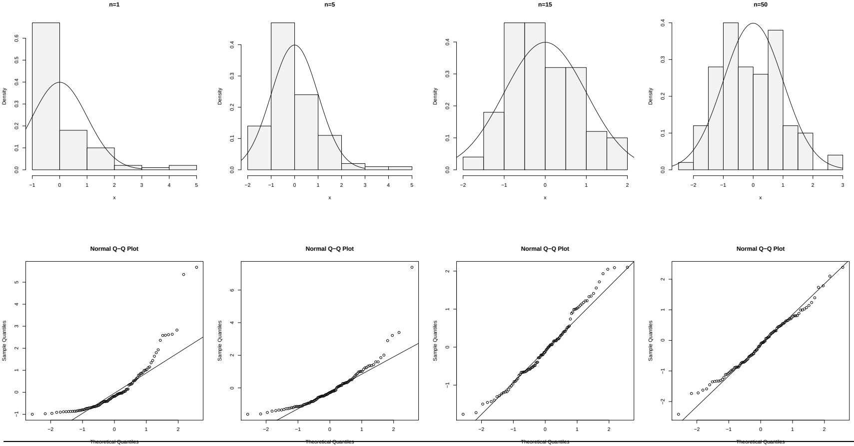
If X^n is a random sample from an **arbitrary** distribution with mean μ and variance σ then the average $\bar{X}^n$ is asymptotically normally distributed, *i.e.*,

$$\bar{X}^n \approx N(\mu, \frac{\sigma^2}{n}) \qquad \text{or} \qquad z = \frac{\bar{X}^n - \mu}{\sigma/\sqrt{n}} \approx N(0,1)$$

- Consequences:
  - allows inference from a sample
  - allows to model errors in measurements: $X = \mu + \epsilon$
- Issues:
  - n should be *enough* large
  - μ and σ must be known



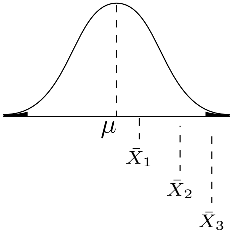
$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$



Inference: Hypothesis Testing and Confidence Intervals

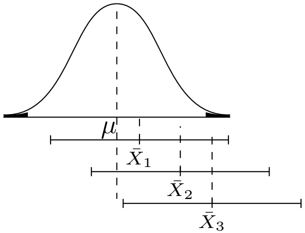
A test of hypothesis determines how likely an sampled estimate $\hat{\theta}$ is to occur under some assumptions on the parameter θ of the population.

$$Pr\left\{\mu - z_1 \frac{\delta}{\sqrt{n}} \leq \bar{X} \leq \mu + z_2 \frac{\delta}{\sqrt{n}}\right\} = 1 - \alpha$$



A confidence interval contains all those values that a parameter θ is likely to assume with probability $1 - \alpha$: $Pr(\hat{\theta}_1 < \theta < \hat{\theta}_2) = 1 - \alpha$

$$Pr\left\{\bar{X} - z_1 \frac{\delta}{\sqrt{n}} \leq \mu \leq \bar{X} + z_2 \frac{\delta}{\sqrt{n}}\right\} = 1 - \alpha$$



And if the variance is unknown...

then we substitute σ with its estimator $\hat{\sigma} = S$

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$

but then

$$z = \frac{X - \mu}{S\sqrt{n}} \approx t_{n-1}$$

i.e., z approximates a t-student distribution.

Terminology

Statistical Hypotheses:

- H_0: null hypothesis
- H_1 alternative hypothesis (one-sided, two-sided)

Within the testing procedure two types of errors are possible:

- error of Type I when the null hypothesis is rejected although it is true
  the level α specifies a priori the assumed likelihood of this error
  α is called level of significance
- error of Type II occurs when a false null hypothesis is not rejected
  it is denoted by β

The power of the test is the likelihood of rejecting a false null hypothesis, $1 - \beta$.

The power of a test depends on the test statistics, the sample size, the alternative hypothesis.

Outline

Two Sample Matched Pairs Case

It corresponds to one run on various instances design

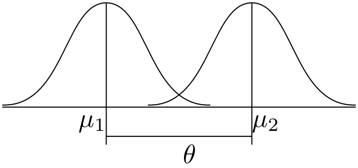
- Student t test `t.test()`
- Wilcoxon test `wilcox.test()`
- Binomial test `binom.test()`
- Permutation solution

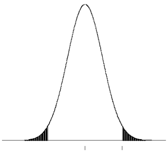
Two Sample Case

It corresponds to a several runs on one single instance design

- Student t test `t.test()`
- Kruskall Wallis `kruskal.test()`
- Permutation solution

Hypothesis Testing Procedure





1. Specify the parameter θ and the test hypothesis, *e.g.*,

$$\theta = \mu_1 - \mu_2 \qquad \begin{cases} H_0 : \theta = 0 \\ H_1 : \theta \neq 0 \end{cases}$$

2. Obtain $P(\theta | \theta = 0)$, the null distribution of θ
3. Compare $\hat{\theta}$ with the upper (in case of one-sided tests) α-quantile of $P(\theta|\theta = 0)$ and accept or reject H_0 according to whether $\hat{\theta}$ is smaller or larger than this value.

Outline

Steps in the Design of Experiments for Algorithms

- Statement of the objectives of the experiment
  - Comparison of different algorithms
  - Impact of algorithm components
  - How instance affect the algorithms
- Identification of the sources of variance
  - Treatment factors, Nuisance factors
  - Factorial Experiment, Experimental unit
  - Complete block design (within-subject matched pairs)
- Definition of test instances. How many instances how many runs?
- Selection of the factor combinations to test
- Running a pilot experiment and refine the design
  - Bugs and no external biases
  - Ceiling or floor effects
  - Rescaling levels of quantitative factors
  - Detect the number of experiments needed to obtained the desired power.

Relevant Experimental Designs

Algorithms $\Rightarrow$ Treatment Factor; Instances $\Rightarrow$ Blocking Factor

Design A: One run on various instances (Unreplicated Factorial)

	Algorithm 1	**Algorithm 2**	...	**Algorithm k**
Instance 1	X_{11}	X_{12}		X_{1k}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Instance b	X_{b1}	X_{b2}		X_{bk}

Design B: Several runs on various instances (Replicated Factorial)

	Algorithm 1	**Algorithm 2**	...	**Algorithm k**
Instance 1	$X_{111}, \ldots, X_{11r}$	$X_{121}, \ldots, X_{12r}$		$X_{1k1}, \ldots, X_{1kr}$
Instance 2	$X_{211}, \ldots, X_{21r}$	$X_{221}, \ldots, X_{22r}$		$X_{2k1}, \ldots, X_{2kr}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Instance b	$X_{b11}, \ldots, X_{b1r}$	$X_{b21}, \ldots, X_{b2r}$		$X_{bk1}, \ldots, X_{bkr}$

Two issues:

- Need to test more than two algorithms
- Need to test more than one factor

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Multisamples – Analysis of Variance

$H_0: \ \mu_1 = \mu_2 = \mu_3 = \dots \qquad H_1: \{\text{at least one differ}\}$

Applying t-test to all pairs the error of Type I is not α but higher:

$$\alpha_{EX} = 1 - \alpha^c$$

Eg, for $\alpha = 0.05$ and $c = 3$ $\alpha_{EX} = 0.14$!

Single Factor Analysis of Variance

$$X_{it} = \mu_i + \epsilon_{it}$$

$$X_{it} = \mu + \alpha_i + \epsilon_{it}$$

$$\sum_{i=1}^{k}\sum_{t=1}^{r}(X_{it} - \bar{X}_{..})^2 = \sum_{i=1}^{k}\sum_{t=1}^{r}(X_{it} - \bar{X}_{i.})^2 + \sum_{i=1}^{k} r(\bar{X}_{i.} - \bar{X}_{..})^2$$

(decomposition in *within-group* and *between-group* sum of squares)

$$MST = \frac{\sum_{i=1}^{k} r\left(\bar{X}_{i.} - \bar{X}_{..}\right)^2}{k-1} \qquad MSE = \frac{\sum_{i=1}^{k}\sum_{t=1}^{r}\left(X_{it} - \bar{X}_{i.}\right)^2}{N-k}$$

(MST *mean square per treatment* and MSE *mean square per error*)

$$F = \frac{MST}{MSE}$$

$F \sim F_{k-1,r-k}$, *i.e.*, the F-ratio approximates a the Fisher distribution F_{df_1,df_2} with df_1 and df_2 degrees of freedom.

Single Factor analysis of Variance (contd.)

- Parametric analysis: ANOVA through F-ratio and Fisher test
  - independent
  - normally distributed
  - homoschedastic
- Nonparametric analysis: Kruskall Wallis
  Assumptions
  - independent
  - homoschedastic

Two Factors Analysis of Variance

Model:

$$X_{hi} = \mu + \alpha_i + \theta_h + \epsilon_{hi}$$

Mean square:

$$MST = \frac{b \sum_{i=1}^{k} (\bar{X}_{.i} - \bar{X}_{..})^2}{k-1}; \quad MSE = \frac{\sum_{h=1}^{b} \sum_{i=1}^{k} \left(X_{hi} - \bar{X}_{h.} - \bar{X}_{.i} + \bar{X}_{..} \right)^2}{bk - b - k + 1}$$

Statistical Tests

- ANOVA through F-ratio and Fisher test
- Friedman test

Two Factors Repeated Measures Analysis of Variance

Model:

$$X_{hit} = \mu + \alpha_i + \theta_h + \epsilon_{hit}$$

or alternatively:

$$X_{hit} = \mu + \alpha_i + \theta_h + \alpha\theta_{hi} + \epsilon_{hit}$$

Statistical Tests

- ANOVA through F-ratio and Fisher test
- Friedman test

If the interactions between factors are of interest interaction plots are useful to visualize them