LOCAL SEARCH METHODS
APPLICATIONS AND ENGINEERING

Lecture 4

Search Landscape (continued) and 'Simple' Local Search Methods

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Outline

1. Plateaux
2. Barriers and Basins
3. Construction Heuristics and Perturbative Searches for SMTWTP and GCP
4. Iterative Improvement Extensions

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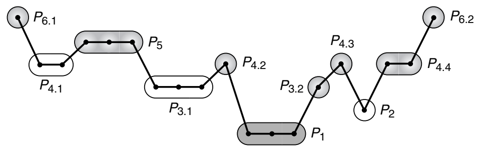
Plateaux

Plateaux, *i.e.*, 'flat' regions in the search landscape, are characteristic for the neutral landscapes obtained for combinatorial problems such as SAT.

Intuition: Plateaux can impede search progress due to lack of guidance by the evaluation function.



Definitions

- **Region:** connected set of search positions.
- **Border of region R:** set of search positions with at least one direct neighbour outside of R (**border positions**).
- **Plateau region:** region in which all positions have the same level, *i.e.*, evaluation function value, l.
- **Plateau:** maximally extended plateau region, *i.e.*, plateau region in which no border position has any direct neighbours at the plateau level l.
- **Solution plateau:** Plateau that consists entirely of solutions of the given problem instance.
- **Exit of plateau region R:** direct neighbour s of a border position of R with lower level than plateau level l.
- **Open / closed plateau:** plateau with / without exits.

Measures of plateau structure:

- *plateau diameter* = diameter of corresponding subgraph of G_N
- *plateau width* = maximal distance of any plateau position to the respective closest border position
- *plateau branching factor* = fraction of neighbours of a plateau position that are also on the plateau.
- *number of exits*, *exit density*
- *distribution of exits within a plateau*, *exit distance distribution* (in particular: avg./max. distance to closest exit)

Some plateau structure results for SAT:

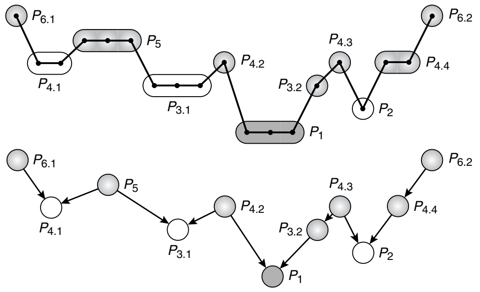
- Plateaux typically don't have an interior, *i.e.*, almost every position is on the border.
- The diameter of plateaux, particularly at higher levels, is comparable to the diameter of search space. (In particular: plateaux tend to span large parts of the search space, but are quite well connected internally.)
- For open plateaux, exits tend to be clustered, but the average exit distance is typically relatively small.

Idea: Obtain abstract view of neutral landscape by collapsing positions on the same plateau into 'macro positions'.

Plateau connection graphs (PCGs):

- *Vertices:* plateaux of given landscape
- *Edges (directed):* connect plateaux that are directly connected by one or more exit.
- Additionally, *edge weights* can be used to indicate the relative numbers of exits from one plateau to its PCG neighbours.

Example: Simple landscape L and plateau connection graph PCG...



Note: The plateaux form a partition of L, *i.e.* every position in L is part of exactly one (possibly degenerate) plateau.

Outline

Barriers and Basins

Observation:

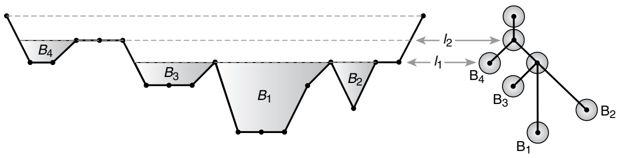
The *difficulty of escaping* from closed plateaux or strict local minima is related to the *height of the barrier*, *i.e.*, the difference in evaluation function, that needs to be overcome in order to reach better search positions:

Higher barriers are typically more difficult to overcome (this holds, *e.g.*, for Probabilistic Iterative Improvement or Simulated Annealing).

Definitions:

- Positions s, s' are *mutually accessible at level* l iff there is a path connecting s' and s in the neighbourhood graph that visits only positions t with $g(t) \leq l$.
- The *barrier level between positions* s, s', $bl(s, s')$ is the lowest level l at which s' and s' are mutually accessible; the difference between the level of s and $bl(s, s')$ is called the *barrier height between* s *and* s'.
- The *depth of a position* s is the minimal barrier height between s and any position s' at a level lower than s, *i.e.*, for which $g(s') < g(s)$.
- **Basins**, *i.e.*, maximal (connected) regions of search positions below a given level, form an important basis for characterising search space structure.

Example: Basins in a simple search landscape and corresponding basin tree

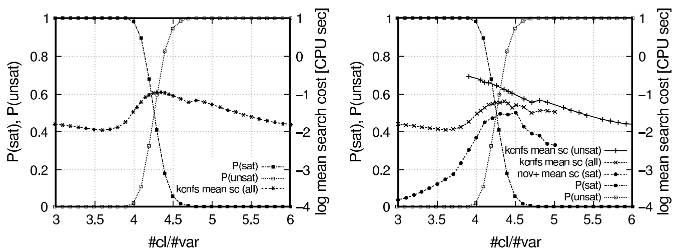


Note: The basin tree only represents basins just below the critical levels at which neighbouring basins are joined (by a *saddle*).

Note:

- Like plateau connection graphs, basin trees can provide much deeper insights into SLS behaviour and problem hardness than global measures of search space structure, such as FDC or ACC.
- **But:** This type of analysis is computationally expensive, since it requires enumeration (or sampling) of large parts of the search space.

Phase Transition for 3-SAT



Outline

Single Machine Total Weighted Tardiness Problem

Construction Heuristics

- Earliest due date
- Modified due date
- Apparent urgency

Iterative Improvement

- Interchange (size $n(n-1)/2$; delta computation in $O(|i-j|)$; complete examination $O(n^3)$)
- Insert (size $(n-1)^2$; delta computation in $O(|i-j|)$; complete examination $O(n^3)$)
- Swap (size $n-1$; delta computation in $O(1)$; complete examination $O(n)$)

Speed-ups:

- Interchange fast branching
- Insert exploration as sequence of swaps

Graph Colouring Problem

Construction Heuristics

- Greedy Algorithm. Vertex order decided:
  - static order (random, largest degree, smallest last degree)
  - dynamic order (largest saturation first)
- Recursive Largest First

Iterative Improvement

Three approaches:

- k fixed, complete (improper) colourings ($g(s) = |E^c|$ or $g(s) = |V^c|$)
  - One exchange
  - Two exchange (but quadratic complexity, hence less used)
- k fixed, partial (proper) colourings ($g(s) = \sum_{v \in C_{imp}} d_{C_{imp}}(v)$)
  - i-swap
- k variable, complete colourings ($g(s) = -\sum_{i=1}^{k} |C_i| + 2 \sum_{i=1}^{k} |C_i||E_i|$)
  - Kempe Chains

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Variable Neighborhood Descent

- *Recall:* Local minima are relative to neighborhood relation.
- **Key idea:** To escape from local minimum of given neighbourhood relation, switch to different neighbhourhood relation.
- Use k neighbourhood relations $N_1, \ldots, N_k$, (typically) ordered according to increasing neighbourhood size.
- Always use smallest neighbourhood that facilitates improving steps.
- Upon termination, candidate solution is locally optimal w.r.t. all neighbourhoods

Variable Neighbourhood Descent (VND):

determine initial candidate solution s
$i := 1$
Repeat:
  choose a most improving neighbour s' of s in N_i
  If $g(s') < g(s)$:
    $s := s'$
    $i := 1$
  Else:
    $i := i + 1$
Until $i > k$

Note:

- VND often performs substantially better than simple II or II in large neighbourhoods [Hansen and Mladenović, 1999]
- Many variants exist that switch between neighbhourhoods in different ways.
- More general framework for LS algorithms that switch between multiple neighbourhoods: *Variable Neighbourhood Search (VNS)* [Mladenović and Hansen, 1997].

Variable Depth Search

- **Key idea:** *Complex steps* in large neighbourhoods = variable-length sequences of *simple steps* in small neighbourhood.
- Use various *feasibility restrictions* on selection of simple search steps to limit time complexity of constructing complex steps.
- Perform Iterative Improvement w.r.t. complex steps.

Variable Depth Search (VDS):

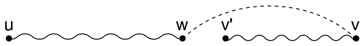
determine initial candidate solution s
$\hat{t} := s$
While s is not locally optimal:
  Repeat:
    select best feasible neighbour t
    If $g(t) < g(\hat{t})$: $\hat{t} := t$
  Until construction of complex step has been completed
  $s := \hat{t}$

Example: The Lin-Kernighan (LK) Algorithm for the TSP (1)

- Complex search steps correspond to sequences of 2-exchange steps and are constructed from sequences of *Hamiltonian paths* (= paths that visit every node in given graph exactly once).
- *δ-path:* Hamiltonian path p + 1 edge connecting one end of p to interior node of p ('lasso' structure):

Basic LK exchange step:

- Start with Hamiltonian path $(u, \ldots, v)$:
- Obtain δ-path by adding an edge (v, w):
- Break cycle by removing edge (w, v'):



- *Note:* Hamiltonian path can be completed into Hamiltonian cycle by adding edge (v', u):

Construction of complex LK steps:

1. start with current candidate solution (Hamiltonian cycle) s; set $t^* := s$; set $p := s$
2. obtain δ-path p' by replacing one edge in p
3. consider Hamiltonian cycle t obtained from p by (uniquely) defined edge exchange
4. if $w(t) < w(t^*)$ then set $t^* := t$; $p := p'$; go to step 2
5. else accept t^* as new current candidate solution s

Note: This can be interpreted as sequence of 1-exchange steps that alternate between δ-paths and Hamiltonian cycles.

Additional mechanisms used by LK algorithm:

- *Tabu restriction:* Any edge that has been added cannot be removed and any edge that has been removed cannot be added in the same LK step.
  Note: This limits the number of simple steps in a complex LK step.
- *Limited form of backtracking* ensures that local minimum found by the algorithm is optimal w.r.t. standard 3-exchange neighbourhood
- (For further details, see text book)

Note:

Variable depth search algorithms have been very successful for other problems, including:

- the Graph Partitioning Problem [Kernigan and Lin, 1970];
- the Unconstrained Binary Quadratic Programming Problem [Merz and Freisleben, 2002];
- the Generalised Assignment Problem [Yagiura *et al.*, 1999].