►►►9.7. Properties of conditional expectation: a list

These properties are proved in Section 9.8. All X's satisfy $\mathsf{E}(|X|) < \infty$ in this list of properties. Of course, $\mathcal{G}$ and $\mathcal{H}$ denote sub-σ-algebras of $\mathcal{F}$. (The use of 'c' to denote 'conditional' in (cMON), etc., is obvious.)

(a) If Y is any version of $\mathsf{E}(X|\mathcal{G})$ then $\mathsf{E}(Y) = \mathsf{E}(X)$. (*Very* useful, this.)

(b) If X is $\mathcal{G}$ measurable, then $\mathsf{E}(X|\mathcal{G}) = X$, a.s.

(c) **(Linearity)** $\mathsf{E}(a_1X_1 + a_2X_2|\mathcal{G}) = a_1\mathsf{E}(X_1|\mathcal{G}) + a_2\mathsf{E}(X_2|\mathcal{G})$, a.s.
Clarification: if Y_1 is a version of $\mathsf{E}(X_1|\mathcal{G})$ and Y_2 is a version of $\mathsf{E}(X_2|\mathcal{G})$, then $a_1Y_1 + a_2Y_2$ is a version of $\mathsf{E}(a_1X_1 + a_2X_2|\mathcal{G})$.

(d) **(Positivity)** If $X \geq 0$, then $\mathsf{E}(X|\mathcal{G}) \geq 0$, a.s.

(e) **(cMON)** If $0 \leq X_n \uparrow X$, then $\mathsf{E}(X_n|\mathcal{G}) \uparrow \mathsf{E}(X|\mathcal{G})$, a.s.

(f) **(cFATOU)** If $X_n \geq 0$, then $\mathsf{E}[\liminf X_n|\mathcal{G}] \leq \liminf \mathsf{E}[X_n|\mathcal{G}]$, a.s.

(g) **(cDOM)** If $|X_n(\omega)| \leq V(\omega)$, $\forall n$, $\mathsf{E}V < \infty$, and $X_n \to X$, a.s., then

$$\mathsf{E}(X_n|\mathcal{G}) \to \mathsf{E}(X|\mathcal{G}), \quad \text{a.s.}$$

(h) **(cJENSEN)** If $c : \mathbf{R} \to \mathbf{R}$ is convex, and $\mathsf{E}|c(X)| < \infty$, then

$$\mathsf{E}[c(X)|\mathcal{G}] \geq c(\mathsf{E}[X|\mathcal{G}]), \quad \text{a.s.}$$

Important corollary: $\|\mathsf{E}(X|\mathcal{G})\|_p \leq \|X\|_p$ for $p \geq 1$.

(i) **(Tower Property)** If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathsf{E}[\mathsf{E}(X|\mathcal{G})|\mathcal{H}] = \mathsf{E}[X|\mathcal{H}], \quad \text{a.s.}$$

Note. We shorthand LHS to $\mathsf{E}[X|\mathcal{G}|\mathcal{H}]$ for tidiness.

(j) **('Taking out what is known')** If Z is $\mathcal{G}$-measurable and bounded, then

$$(*) \qquad \mathsf{E}[ZX|\mathcal{G}] = Z\mathsf{E}[X|\mathcal{G}], \quad \text{a.s.}$$

If $p > 1$, $p^{-1} + q^{-1} = 1$, $X \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathsf{P})$ and $Z \in \mathcal{L}^q(\Omega, \mathcal{G}, \mathsf{P})$, then (*) again holds. If $X \in (\mathrm{m}\mathcal{F})^+$, $Z \in (\mathrm{m}\mathcal{G})^+$, $\mathsf{E}(X) < \infty$ and $\mathsf{E}(ZX) < \infty$, then (*) holds.

(k) **(Rôle of independence)** If $\mathcal{H}$ is independent of $\sigma(\sigma(X), \mathcal{G})$, then

$$\mathsf{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathsf{E}(X|\mathcal{G}), \quad \text{a.s.}$$

In particular, if X is independent of $\mathcal{H}$, then $\mathsf{E}(X|\mathcal{H}) = \mathsf{E}(X)$, a.s.