Exercises for MM513

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In the following $\Omega, \mathcal{F}, P)$ denotes a probability space.. If X is an sv on $(\Omega, \mathcal{F}, P)$, then we let $X(P)$ denote the image measure of X, i.e. the distribution of X. If μ og ν are two probability measures on $\mathbb{R}$, we let $\mu \otimes \nu$ denote the product measure of μ and ν.

Problem 1

Let $\mathcal{G}$ be a sub–σ–algebra of$\mathcal{F}$. A subset $B \in \mathcal{G}$ with $P(B) > 0$ is called an atom in $\mathcal{G}$ if we have:

$$\forall A \in \mathcal{G} : A \subseteq B \Rightarrow P(A) = 0 \vee P(A) = P(B).$$

(i) Show that if Y is a $\mathcal{G}$–measurable stochastic variable and B is an atom in $\mathcal{G}$, then Y is almost surely constant on B.

(ii) Show that if A og B are atoms in $\mathcal{G}$ with $P(A \setminus B) > 0$, then $P(A \cap B) = 0$. Hence except for the zero set $A \cap B$ A and B are disjoint.

Problem 2

In this problem we let $\mathcal{G}$ denote a finite sub–σ–algebra of $\mathcal{F}$.

(i) Showe that if $A \in \mathcal{G}$ med $P(A) > 0$, then there exists an atom B in $\mathcal{G}$ with $B \subseteq A$.

(ii) Show that if $A \in \mathcal{G}$ med $P(A) > 0$, then:

$$A = \cup\{B \subseteq A \mid B \text{ atom i } \mathcal{G}\}.$$

Note that it is a finite union of sets.

(iii) Let $\{B_j \mid 1 \leq j \leq n\}$ be a maximal set of atoms in $\mathcal{G}$. According to Problem 1 (ii) we are talking about all atoms in $\mathcal{G}$ except for adding or removing sets of measure 0.

Let $X \in L_1(P)$. Show that

$$E(X \mid \mathcal{G}) = \sum_{j=1}^{n} \frac{1}{P(B_j)} \int_{B_j} X dP \quad 1_{B_j}.$$

Hint: Show that the right hand side satisfies the usual integral equation for conditional expectations. Use also that according to (ii) every set in $\mathcal{G}$ is the union of some of the B_j'erne.

Problem 3

If Y is a stochastic variable,then we let as usual $\sigma(Y)$ denote the smallest σ–algebra in which Y is measurable. If $X, Y \in L_1(P)$ we put $E(X \mid Y) = E(X \mid \sigma(Y))$.

(i) Show that if $X, Y \in L_1(P)$ are independent, then $E(X \mid Y) = E(X)$.

Hint: Use that $\omega \to E(X)$ er $\mathcal{G}$–measurable.

(ii) Find an example of stochastic variables X and Y so that $E(X \mid Y) = E(X)$, but X and Y are not independent.

Problem 4

Let X and Y be real stochastic variables.

(i) Show that X og Y are independent if and only if

$$E(f(X)g(Y)) = E(f(X))E(g(Y))$$

for all bounded Borel functions $f, g : \mathbb{R} \to \mathbb{R}$.

Hint to the "if" part: Put $f = 1_A$ and $g = 1_B$, where A and B are arbitrary Borel sets.

(ii) Show that X and Y are independent if and only if

$$E(f(X) \mid Y) = E(f(X))$$

l for all bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$.

problem 5

Let $1 \leq p < \infty$ and let $\mathcal{G}$ be a sub–σ–algebra of $\mathcal{F}$.

- Let $X \in L_p(P)$. Use Jensen's inequality to show that $E(X \mid \mathcal{G}) \in L_p(P)$ and that $\|E(X \mid \mathcal{G})\|_p \leq \|X\|_p$.

(ii) Show that if $X \in L_\infty(P)$, then $E(X \mid \mathcal{G}) \in L_\infty(P)$ with$\|E(X \mid \mathcal{G})\|_\infty \leq \|X\|_\infty$

(ii) Lad now $1 \leq p \leq \infty$ and let $(X_n) \subseteq L_p(P)$ and $X \in L_p(P)$, so that $X_n \to X$ in $L_p(P)$. Show that $E(X_n \mid \mathcal{G}) \to E(X \mid \mathcal{G})$ in $L_p(P)$. Hence the Operation $E(\cdot \mid \mathcal{G})$ is a continuous operation in $L_p(P)$.

Problem 6

Let $(\mathcal{F}_n)$ be a filtering of $\mathcal{F}$, let (X_n) be a submartingale relative to $(\mathcal{F}_n)$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function.

(i) Use Jensen's inequality to show that if $\phi(X_n) \in L_1(P)$ for all $n \in \mathbb{N}$ and ϕ is increasing, then $(\phi(X_n))$ is a submartingale. Show next that if (X_n) is a martingale, then the conclusion holds without the assumption that ϕ is increasing.

(ii) Let $1 \leq p < \infty$ and assume that $X_n \in L_p(P)$ for all $n \in \mathbb{N}$and that (X_n) is a martingale. Show that $|X_n|^p$ is a submartingale.

(iii) If $x \in \mathbb{R}$, we put $x^+ = x$ if $x \geq 0$ and $x^+ = 0$ if $x < 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = x^+$ for all $x \in \mathbb{R}$. Argue (i.e by the using a drawing) that ϕ is convex. Show next that (X_n^+) is a submartingale.

Problem 7

Let$(X_n) \subseteq L_1(P)$ be a sequence of independent, identically distributed random variables with mean value 0 and variance σ^2. Define:

$$Y_n = (\sum_{k=1}^{n} X_k)^2$$

and

$$Z_n = Y_n - n\sigma^2$$

for all $n \in \mathbb{N}$. Put for every n $\mathcal{F}_n = \sigma(X_k \mid 1 \leq k \leq n)$.

Show that (Y_n) is a submartingale and that (Z_n) is a martingale.

Problem 8

Let $(\mathcal{F}_n)$ be a filtration of $\mathcal{F}$ and let $X = (X_n)$ be an $(\mathcal{F}_n)$–adapted process. (X_n) is called a local martingale if there exists a sequence (T_k) of finite stopping times with $T_k \uparrow \infty$ for $k \to \infty$ and so that $(X_{n \wedge T_k})$ is a martingale for every $k \in \mathbb{N}$. Such processes are often seen in mathematical finance. Show that a downwards bounded local (X_n) is a som er nedadtil begrænset, er en supermartingale. Hint: Use Fatou's lemma in a suitable way.

Problem 9

Let $(\mathcal{F}_n)$ be a filtration of $\mathcal{F}$ and let $X = (X_n)$ be an $(\mathcal{F}_n)$–adapted process so that $X_{n+1} - X_n$ is independent of $\mathcal{F}_n$ for all $n \in \mathbb{N}$.

(i) Assume that $E(X_n) = E(X_1)$ for all $n \in \mathbb{N}$. Show (X_n) is a martingale.

(ii) Assume that $(E(X_n))$ is an increasing sequence. Show that (X_n) is a submartingale.

(iii) Guess the next question yourselves!!!

Problem 10

Let (r_n) be a sequence of independent stochastic variables with $P(r_n = 1) = P(r_n = -1) = \frac{1}{2}$ and put $\mathcal{F}_n = \sigma(r_j; 1 \leq j \leq n)$ for all $n \in \mathbb{N}$.

(i) Show that (r_n) is an orthonormal sequence in $L_2(P)$.

(ii) Let $(t_n) \in \mathbb{R}$ be an arbitrary sequence and put for every $n \in \mathbb{N}$ $S_n = \sum_{k=1}^{n} t_k r_k$. Show that (S_n) is a martingale.

(iii) Prove that the following statements are equivalent:

1. $\sum_{k=1}^{\infty} t_k^2 < \infty$.
2. The series $S_\infty = \sum_{k=1}^{\infty} t_k r_k$ converges in $L_2(P)$.
3. The seies $S_\infty = \sum_{k=1}^{\infty} t_k r_k$ converges in $L_1(P)$.
4. The series $S_\infty = \sum_{k=1}^{\infty} t_k r_k$ converges a.s.

(iv) Show that if one of the conditions (and hence all) is satisfied, then:

$$(\sum_{k=1}^{\infty} t_k^2)^{\frac{1}{2}} = \|S_\infty\|_2$$

Problem 11

Let (r_n) be as in Problem 10.

(i) Show that if $(t_k) \subseteq \mathbb{R}$ og $1 \leq p \leq 2$, then we have:

$$\|\sum_{k=1}^{n} t_k r_k\|_p \leq (\sum_{k=1}^{n} t_k^2)^{\frac{1}{2}} \quad \text{for alle } n \in \mathbb{N}.$$

The aim of the rest of the problem is to prove that there is a constant $A_1 > 0$, so that:

$$A_1(\sum_{k=1}^{n} t_k^2)^{\frac{1}{2}} \leq \|\sum_{k=1}^{n} t_k r_k\|_1$$

for all $(t_k) \subseteq \mathbb{N}$ and all $n \in \mathbb{N}$.

We assume from now on that such a constant A_1 does not exist and wish to reach a contradiction.

(ii) Shw that the assumption implies that for every $K > 0$ there exist an $n \in \mathbb{N}$ and $(t_k)_{k=1}^{n} \subseteq \mathbb{R}$ so that:

$$\|\sum_{k=1}^{n} t_k r_k\|_1 \leq 1 \quad \sum_{k=1}^{n} t_k^2 \geq K$$

(iii) Put $p_0 = 0$. Show by induction (and with the help of (ii)) that there exist a stricly increasing sequence $(p_n) \subseteq \mathbb{N}$ and a sequence $(s_k) \subseteq \mathbb{R}$, so that:

$$\sum_{k=p_n+1}^{p_{n+1}} s_k^2 \geq 2^{2n} \quad \text{for alle } n \geq 0$$

and

$$\|\sum_{k=p_n+1}^{p_{n+1}} s_k r_k\|_1 \leq 1 \quad \text{for alle } n \geq 0.$$

(iv) For every $n \geq 0$ and every $p_n < k \leq p_{n+1}$ we put $t_k = 2^{-n} s_k$. Show that $\sum_{n=0}^{\infty} \sum_{k=p_n+1}^{p_{n+1}} t_k r_k$ is convergent in $L_1(P)$ while $\sum_{k=1}^{\infty} t_k^2 = \sum_{n=0}^{\infty} \sum_{k=p_n+1}^{p_{n+1}} t_k^2$ is divergent.

(v) Explain why this is in contradiction toProblem 10. Hence we have proved the existence of A_1.

(vi) Show that for all $1 \leq p \leq 2$, all $n \in \mathbb{N}$, and all $(t_k) \subseteq \mathbb{R}$ we have:

$$A_1 (\sum_{k=1}^{n} t_k^2)^{\frac{1}{2}} \leq \|\sum_{k=1}^{n} t_k r_k\|_p \leq (\sum_{k=1}^{n} t_k^2)^{\frac{1}{2}}.$$

This inequality is called Khintchine's inequality (for $1 \leq p \leq 2$). By a duality argument one can get an analoguous inequality for $2 < p < \infty$. It has been proved by Uffe Haagerup that the best choice of the constant A_1 is $A_1 = \frac{1}{\sqrt{2}}$. I do not know whether this can be shown using martingale theory.

Problem 12

Let$(M_n)_{n \geq 0}$ be martingale which is bounded in $L_2(P)$ and define M_∞ as in the notes. Show that $E(M_\infty \mid \mathcal{F}_n) = M_n$ for alle $n \geq 0$.

Problem 13

Let $(X_n) \subseteq L_2(P)$ be a sequence of independent stochastic variables with $E(X_n) = 0$ for all$n \in \mathbb{N}$. Put $X_0 = 0$ and let as usual $\mathcal{F}_n = \sigma\{X_k \mid 0 \leq k \leq n\}$. In addition we put $S_0 = 0$ and

$$S_n = \sum_{k=1}^{n} X_k \quad \text{for alle } n \in \mathbb{N}.$$

From the book and the notes it follows that $S = (S_n)$ is a martingale.

(i) Show that for all $n \in \mathbb{N}$ we have:

$$S_n^2 - S_{n-1}^2 = X_n S_n + X_n S_{n-1}.$$

(ii) Find the Doob decomposition of S^2. Hint: One can e.g. use the formula in the notes to find (A_n).

(iii) If one does not know the Doob decompostion, one can attack it as follows (and do that!!): Calculate $E(S_n^2 \mid \mathcal{F}_{n-1})$ for all $n \in \mathbb{N}$. Use this expression to find what has to be subtracted from S_n^2 to get a martingale.

Problem 14

Let $(X_n))$ be an $L_2(P)$–bounded martingale and put

$$X_\infty = \lim X_n.$$

This limit exists a.s. and in $L_2(P)$ according to the notes. Show that (X_n^2) is a uniformly integrable submartingale and that

$$X_\infty^2 = \lim X_n^2 \quad \text{a.s. and in } L_1(P).$$

Hint: Show first that $E(X_\infty^2 \mid \mathcal{F}_n) \geq X_n^2$ a.s.

Problem 15

This problem is a generalization of Theorem 3.2 of the notes. Let $1 < p < \infty$ and let (X_n) be a martingale which is bounded $L_p(P)$, e.g. $\sup_n E(|X_n|^p) < \infty$.

(i) Show that $X_\infty = \lim X_n$ exists a.s. and in $L_1(P)$.

(ii) Show that $E(X_\infty \mid \mathcal{F}_n) = X_n$ for all $n \in \mathbb{N}$.

(iii) Show that $X_\infty \in L_p(P)$.

(iv) Show that $E(|X_\infty|^p \mid \mathcal{F}_n) \geq |X_n|^p$ for all $n \in \mathbb{N}$ and conclude that $(|X_n|^p)$ is uniformly integrable.

(v) Use the convexity of $|\cdot|^p$ to show that

$$|X_\infty - X_n|^p \leq 2^{p-1}(|X_\infty|^p + |X_n|^p) \quad \text{for alle } n \in \mathbb{N}$$

and conclude that $(|X_\infty - X_n|^p)$ is uniformly integrable. Show next that $X_\infty = \lim X_n$ i $L_p(P)$.

Problem 16

Let $(X_n) \subseteq L_1(P)$ be a sequence of independent stochastic variables and put $X_0 = 0$, $S_0 = 0$, and $S_n = \sum_{k=1}^n X_k$ for all $n \in \mathbb{N}$. As usual we let $\mathcal{F}_n = \sigma\{X_k \mid 1 \leq k \leq n\}$ for all $n \in \mathbb{N}$. Find the Doob decomposition of (S_n).

Problem 17

Let $(\mathcal{F}_n)$ be a filtration of $\mathcal{F}$ so that all P–zero sets belongs $\mathcal{F}_0$ and let τ be a stopping time. In addition we let $\mathcal{F}_\tau$ the subset of $\mathcal{F}$ consisting of all those $A \in \mathcal{F}$, for which $A \cap (\tau = n) \in \mathcal{F}_n$ for all $n \geq 0$.

1. Show that $\mathcal{F}_\tau$ is a σ–algebra.
2. Show that if σ is a stopping time with $\sigma \leq \tau$ n.s., then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
3. Let $(X_n)_{n\geq 0}$ be a process which is adapted the filtration and assume that $\tau < \infty$ a.s. Show that X_τ is $\mathcal{F}_\tau$–measurable.
4. Assume in addition that $(X_n) \subseteq L_1(P)$ and that there exists an $M \in \mathbb{N}$, so that $\tau \leq M$ a.s. Show that
$$|X_\tau| \leq \sum_{n=0}^{M} |X_n|,$$
and conclude that $X_\tau \in L_1(P)$

In the following we let $(\mathcal{F}_n)$ be a filtration which satisfies the condition in Problem 17.

Problem 18 (optional sampling)

Let (X_n) be a submartingale (with respect to $(\mathcal{F}_n)$), and let σ and τ be bounded stopping times with $\sigma \leq \tau$ a.s.

1. Show that if $m < k$ and $A \in \mathcal{F}_\sigma$, then
$$\int_{A\cap(\sigma=m)} X_k dP \geq \int_{A\cap(\sigma=m)} X_m dP.$$
2. Show that $E(X_\tau \mid \mathcal{F}_\sigma) \geq X_\sigma$. (Hint: Write $X_\tau - X_\sigma$ as a martingale transform with a suitable C and use this to prove that if$m \geq 0$, then $\int_{A\cap(\sigma=m)}(X_\tau - X_\sigma)dP \geq 0$; hereafter sum over m.)

The corresponding result for supermartingales shows, that it is not possible to turn a non–favorable play to a favorable one by using bounded stopping times.

Let $0 < p \leq 1$ and let $(X_n)_{n\geq 1}$ be a sequence of independent stochastic variables so that $P(X_n = 1) = p$ and $P(X_n = -1) = 1 - p$ for all $n \in \mathbb{N}$. If $a \in \mathbb{R}$, we put $X_0 = a$ and $S_n = \sum_{k=0}^{n} X_k$ and let in this case $\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$. Compare this to the beginning of Section 2 of the notes. (S_n) is called a simple random walk with parameter p and starting at a. If $p = \frac{1}{2}$, (S_n) is called symmetric.

Problem 19

Let $a, k \in \mathbb{N}$ with $a < k$, let (S_n) be a simple symmetric random walk starting at a. It follows from earlier results that (S_n) is a martingale. Further we let

$$\tau = \inf\{n \geq 1 \mid S_n = 0 \quad \text{or } S_n = k\}.$$

It follows from Proposition 2.6 in the notes that τ is a stopping time. It can be proved that $P(\tau < \infty) = 1$.

1. Show that (S_n) og τ satisfies the conditions in Problem 27.
2. Show that $E(S_\tau) = a$.
3. Show that $P(S_\tau = k) = \frac{a}{k}$. (Hint: Split $E(S_\tau)$ as the sum of the integral over the set, where $S_\tau = 0$ and the integral over the set, where $S_\tau = k$.)

Note that $P(S_\tau = k)$ gives the probability that you get k kroner out of your game, before you get bankrupt (i.g.$S_\tau = 0$)!!

Problem 20

Let $0 < p < 1, p \neq \frac{1}{2}$, let $a, k \in \mathbb{N}$ with $a < k$, and let in addition (S_n) be a simple random walk with parameter p, starting at a. Further, let (X_n) be defined as above and put

$$Z_n = (\frac{1-p}{p})^{S_n} \quad \text{for all } n \in \mathbb{N},$$

and let

$$\tau = \inf\{n \geq 1 \mid S_n = 0 \quad \text{or } S_n = k\}$$

1. Show that $E((\frac{1-p}{p})^{X_n}) = 1$ for all $n \geq 1$ and conclude that (Z_n) is a martingale.
2. Show that $E(Z_\tau) = (\frac{1-p}{p})^a$.
3. Show that $P(S_\tau = k) = P(Z_\tau = (\frac{1-p}{p})^k) = \frac{1-(\frac{1-p}{p})^a}{1-(\frac{1-p}{p})^k}$.

Problem 21

Let $X \in L_1(P))$ and let $\mathcal{G}$ and $\mathcal{H}$ be sub σ algebras of $\mathcal{F}$. Let further $\mathcal{H}$ be independent of $\sigma(X, \mathcal{G})$. The aim of this problem is to prove that

$$E(X \mid \sigma(\mathcal{G}, \mathcal{H})) = E(X \mid \mathcal{G}). \tag{1}$$

It is enough to prove (1) for $X \geq 0$. Why? Hence from now on we assume that $X \geq 0$.

1. Let $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Show that

$$\int_{G\cap H} X dP = P(H) \int_G X dP$$

and

$$\int_{G\cap H} E(X \mid \mathcal{G}) dP = P(H) \int_G E(X \mid \mathcal{G}) dP,$$

and conclude that

$$\int_{G\cap H} X dP = \int_{G\cap H} E(X \mid \mathcal{G}) dP.$$

2. Show that

$$\int_A X dP = \int_A E(X \mid \mathcal{G}) dP \quad \text{for all } A \in \sigma(\mathcal{G}, \mathcal{H}).$$

Hint: Use that $\{G \cap H \mid G \in \mathcal{G}, H \in \mathcal{H}\}$ is a suitable generating system for $\sigma(\mathcal{G}, \mathcal{H})$ and use the usual measure theoretical argumentsog benyt de sædvanlige målteoretiske argumenter.

3. Conclude from 2. that (1) holds.

Problem 22

Let X be normally distributed with mean value 0 and variance σ^2. Calculate $E(\exp(X))$ and the variance of $\exp(X)$.

Hint: Use Theorem 5.4 of the notes.

Problem 23

Let (B_n) be a stochastic process satisfying:

(i) $B_0 = 0$ a.s.

(ii) Hvis $0 \leq m < n$, then $B_n - B_m$ is normally distributed with mean value 0 and variance $n - m$.

(iii) If $0 \leq n_1 < n_2 < \cdots < n_k$, then $B_{n_1}, B_{n_2} - B_{n_1}, \cdots, B_{n_k} - B_{n_{k-1}}$ are independent.

(B_n) is called a discreet Brownian motion or a discreet Wiener process. It is not that easy to prove its existence, but this we shall assume as fact here. Note that in (ii) we have $n - m$ og **not** $(n - m)^2$!!

For every $n \in \mathbb{N}$ we put $\mathcal{F}_n = \sigma(B_k, 0 \leq k \leq n)$.

1. Prove that (B_n) is a martingale.

2. Find the Doob decomposition of (B_n^2).

3. Let $a \in \mathbb{R}$, $a \neq 0$ and define:

$$M_n = \exp(aB_n - \frac{1}{2}a^2 n) \quad \text{for all } n \geq 0. \tag{1}$$

   Show that (M_n) is a martingale.

   Hint: Do not use Jensen here! Write for $n \geq 1$

$$M_n = \exp(a(B_n - B_{n-1}) - \frac{1}{2}a^2)M_{n-1},$$

   and use the assumptions and the result from Problem 22.

4. Show that there is a $M_\infty \in L_1(P)$ so that

$$M_n \to M_\infty \quad \text{a.s.}$$

5. Let $\varepsilon > 0$ and put for every $n \geq 0$ $b_n = a^{-1}(\frac{1}{2}a^2 n + \log \varepsilon)$. Determine that $(M_n \geq \varepsilon) = (B_n \geq b_n)$.

6. Show that for $a > 0$ vil

$$M_n \to 0 \quad \text{in probability.}$$

   Conclude from this that $M_\infty = 0$ n.s. Similar calculations can be done for $a < 0$.

7. Is (M_n) uniformly integrable?

Problem 24

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space (μ need not be a finite measure) and let $f : \Omega \to [0, \infty[$ be an $\mathcal{F}$–measurable function. We define ν by:

$$\nu(A) = \int_A f d\mu \quad \text{for all } A \in \mathcal{F}. \tag{1}$$

1. Show that ν is a measure.

2. Let $g : \Omega \to [0, \infty[$ be measurable. Show that

$$\int_\Omega g d\nu = \int_\Omega g f d\mu. \tag{2}$$

3. Let now $g : \Omega \to \mathbb{R}$ be an arbitrary measurable function. Show that $g \in L_1(\nu)$, if and only if $gf \in L_1(\mu)$. Show next that in that case (2) holds.

problem 25

Let $B_n)$, $(\mathcal{F}_n)$, and (M_n) be as defined in Problem 23. In addition, define the process (X_n) by:

$$X_n = B_n - an \quad \text{for all } n \geq 0. \tag{1}$$

In the following we let $N \in \mathbb{N}$ be fixed and put

$$Q(A) = \int_A M_N dP \quad \text{for all } A \in \mathcal{F}. \tag{2}$$

1. Show that Q is a probability measure with the property that for all $A \in \mathcal{F}$ we have that $Q(A) = 0$, if and only if $P(A) = 0$.

2. Show that if $Y \in L_1(Q)$ and Y is $\mathcal{F}_n$–measurable for some n with $0 \leq n \leq N$, then

$$\int_\Omega Y dQ = \int_\Omega Y M_n dP. \tag{3}$$

The aim of the rest of the problem is to prove that $(X_n)_{0 \leq n \leq N}$ is a finite Brownian motion in the probability space $(\Omega, \mathcal{F}, Q)$.

3. Let $0 \leq m < n \leq N$ and let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function. Prove that

$$\int_\Omega f(X_n - X_m) dQ = \tag{4}$$

$$\int_\Omega f(B_n - B_m - a(n-m)) \exp(B_n - B_m - \frac{1}{2}a^2(n-m)) dP =$$

$$(2\pi(n-m)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(u - (n-m)a) \exp(-\frac{(u-(n-m)a)^2}{2(n-m)}) du =$$

$$(2\pi(n-m))^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(u) \exp(\frac{-u^2}{2(n-m)}) du.$$

   Hint: Use (3), that $E(M_m) = 1$, and Theorem 5.4 in the notes.

4. Conclude from 3. that if $0 \leq m < n \leq N$, then $X_n - X_m$ is normally distributed $N(0, n-m)$ in the probability space $(\Omega, \mathcal{F}, Q)$.

   Hint: Let $x \in \mathbb{R}$ and put $f = 1_{]-\infty, x]}$ in (4).

5. Let $0 = n_0 < n_1 < \cdots < n_k \leq N$ and let $x_1, x_2, \cdots, x_k \in \mathbb{R}$. Show that

$$Q(\cap_{j=1}^k (X_j - X_{j-1} \leq x_j)) = \prod_{j=1}^k Q(X_j - X_{j-1} \leq x_j), \tag{5}$$

   and conclude that $X_1, X_{n_2} - X_{n_1}, \cdots, X_{n_k} - X_{n_{k-1}}$ are independent.

   Hint: Prove (5) by induction.ved induktion. In the k'te step it is a good idea to write $M_{n_k} = M_{k-1} \exp(B_{n_k} - B_{n_{k-1}} - \frac{1}{2}a^2(n_k - n_{k-1}))$.

   It has now been proven that $\{X_n \mid 0 \leq n \leq N$ is a finite Brownian motion.

6. Is it possible to make the construction above for the whole sequence $(X_n)_{n\geq 0}$ once and for all? Specifically:Mere specifikt: Does there exist an $M \in L_1(P)$ with $M > 0$ a.s. so that if we put

$$Q(A) = \int_A M dP \quad \text{for all } A \in \mathcal{F},$$

then Q is a probability measure with the property that

$$Q(A) = \int_A M_n dP \quad \text{for all } n \text{ and all } A \in \mathcal{F}_n?$$

Problem 26

Let $(X_n)_{n\geq O} \subseteq L_1(P)$ be a sequence of independent, identically distributed stochastic variables. Put for every $n \geq 0$ $S_n = \sum_{k=0}^{n} X_k$ and $\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$.

1. Show that if $E(X_0) = 0$, then (S_n) is a martingale.
2. Show that if $E(X_0) > 0$, then (S_n) is a submartingale.
3. Guess the next question yourselves!!

Problem 27

Let $(X_n)_{n\geq 0}$ be a martingale relative to the filtration $(\mathcal{F}_n)$ and let τ be a stopping time with $P(\tau < \infty) = 1$. Assume further that there is an M so that således at $|X_n| 1_{(n\leq\tau)} \leq M$, hence that (X_n) is bounded up to the time τ.

1. Show that $|X_\tau| \leq M$ and conclude that $E(|X_\tau|) < \infty$.
2. Show that $E(X_{\tau\wedge n} \to E(X_\tau)$ for $n \to \infty$ and use JP, Theorem 24.2 to conclude that $E(X_\tau) = E(X_0)$.

   (Hint: Write $X_{\tau\wedge n} = X_\tau 1_{(\tau\leq n)} + X_n 1_{(n<\tau)}$.)