Some facts on orthonormal bases of Hilbert spaces

Introduction

The results below can be found with detailed proofs in the notes "Noter om Hilbertrum og indre produkter".

1 Orthonormal bases

Let H be a Hilbert space. A sequence $(x_n) \subseteq H$ of mutual orthogonal vectors (i.e $(x_n, x_m) = 0$ for all $n \neq m$) with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ is called an orthonormal sequence.

Definition 1.1 *An orthonormal sequence (x_n) is called an orthonormal basis for H if for every $x \in H$ there exists a sequence (t_n) of scalars so that*

$$x = \sum_{n=1}^{\infty} t_n x_n.$$

We have the following theorem:

Theorem 1.2 *Let (x_n) be an orthonormal basis for H and let $x \in H$ and (t_n) a sequence of scalars so that $x = \sum_{n=1}^{\infty} t_n x_n$. Then*

(i) $t_n = (x, x_n)$ for all $n \in \mathbb{N}$. In particular the sequence (t_n) is uniquely determined.

(ii $\|x\|^2 = \sum_{n=1}^{\infty} |(x, x_n)|^2$

Proof: (i): Since $x = \sum_{m=1}^{\infty} t_m x_m$, we get from the linearity and the continuity of $(\cdot, \cdot)$, that for all $n \in \mathbb{N}$ we have

$$(x, x_n) = \sum_{m=1}^{\infty} t_m (x_m, x_n) = t_n,$$

which proves (i).

(ii): By (i) $x = \sum_{n=1}^{\infty}(x, x_n)x_n$ and the Pythagoras theorem therefore gives that $\|x\|^2 = \sum_{m=1}^{\infty} |(x, x_n)|^2$ □

In our notes on Hilbert spaces mentioned above the following theorem is proved, using the Gram–Schmidt orthogonalization procedure.

Theorem 1.3 *A Hilbert space has an orthonormal basis if and only if it is separable.*

Since the spaces $L_2(0, 1)$ and $L_2(0, \infty)$ are separable, we get

Corollary 1.4 $L_2(0, 1)$ *and* $L_2(0, \infty)$ *have orthonormal bases.*

In our forthcoming proof of the existence of Brownian motion we will actually' construct concrete orthonormal bases of these two spaces.

The next theorem shows when a given orthonormal sequence (e_n) in H is actually an orthonormal basis

Theorem 1.5 *Let* $(e_n) \subseteq H$ *be an orthonormal sequence. Then we have:*

(i) For all $n \in \mathbb{N}$ *and all* $x \in H$ $x - \sum_{j=1}^{n}(x, e_j)e_j$ *is orthogonal to* $\sum_{j=1}^{n}(x, e_j)e_j$.

(ii) For $n \in \mathbb{N}$ *and all* $x \in H$ $\sum_{j=1}^{n} |(x, e_j)|^2 \leq \|x\|^2$

(iii For all $x \in H$ $\sum_{n=1}^{\infty} |(x, e_n)|^2 \leq \|x\|^2$ *and hence the series* $\sum_{n=1}^{\infty}(x, e_n)e_n$ *is convergent in* H.

(iv) (e_n) *is an orthonormal basis if and only if* $\overline{span((e_n))} = H$.

Proof: (i): If $x \in H$ and $n \in \mathbb{N}$ an easy calculation shows that $(x - \sum_{j=1}^{n}(x, e_j)e_j, \sum_{m=1}^{n}(x, e_m)e_m) = 0$ so that the two vectors are orthogonal.

(ii): From (i) and the Pythagoras theorem it follows that $\|x\|^2 = \|x - \sum_{j=1}^{n}(x, e_j)e_j\|^2 + \sum_{j=1}^{n} |(x, e_j)|^2$ and therefore $\sum_{j=1}^{n} |(x, e_j)|^2 \leq \|x\|^2$.

(iii); Since the inequality in (ii) holds for all $n \in \mathbb{N}$, we get that $\sum_{n=1}^{\infty} |(x, e_n)|^2 \leq \|x\|^2$. Proposition 3.1 of the lecture notes now gives the convergence of the series in H.

(iv): If (e_n) is an orthonormal basis, then every $x \in H$ can be approximated by finite linear combinations of the e_n's and therefore $span((e_n))$ is dense in H. Assume next that the span is dense and let $x \in H$. Since the series in (iii) converges we can put $y = \sum_{n=1}^{\infty}(x, e_n)e_n$. Clearly $(y, e_n) = (x, e_n)$ for all $n \in \mathbb{N}$ and hence $x - y$ is orthogonal to all the e_n's therefore also to the closure of their linear span which is H by assumption. Hence $x = y$, which gives the conclusion. □

2 Extensions of continuous linear maps between normed spaces

The next result is about extensions linear maps between general normed spaces. We recall that a complete normed space is called a Banach space.

Theorem 2.1 *Let X be a normed space, $X_0 \subseteq X$ a dense subspace, and Y a Banach space. Further let $T : X_0 \to Y$ be a linear map so that there is a constant K with $\|Tx\| \leq K\|x\|$ for all $x \in X_0$. Then there is a unique linear map $\tilde{T} : X \to Y$ so that $\|\tilde{T}x\| \leq K\|x\|$ for all $x \in X$ and $\tilde{T}x = Tx$ for all $x \in X_0$*

Note that the condition on T is equivalent to the continuity of T.

Proof: Let $x \in X$. Since X_0 is dense in X, we can find a sequence $(x_n) \subseteq X_0$ with $x_n \to x$ for $n \to \infty$. Since T is linear, we have that

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq K\|x_n - x_m\| \quad \text{for all } n, m \in \mathbb{N}. \tag{2.1}$$

Since (x_n) is convergent and hence a Cauchy sequence, equation (2.1) shows that (Tx_n) is a Cauchy sequence in Y, and since Y is complete, it is convergent in Y. Put $y = \lim Tx_n$. We wish to show that y only depends on x and not on (x_n) so that we can define $\tilde{T}x = y$. Note that we have actually shown that everytime we take a sequence from X_0 converging to x, then the image sequence will converge in Y.

Let now $(z_n) \subseteq X_0$ be another sequence converging to x and put $z = \lim Tz_n$. Define a new sequence $(u_n) \subseteq X_0$ by $u_{2n-1} = x_n$ and $u_{2n} = z_n$ for all $n \in \mathbb{N}$. Clearly also $u_n \to x$ and therefore $\lim Tu_n$ exists in Y. However, since both (Tx_n) and (Tz_n) are subsequences of (Tu_n), we must have $z = \lim Tu_n = y$. Hence we can define $\tilde{T}x = \lim Tx_n$ and since the limit does not depend on the actual sequence (x_n), it is also clear that $\tilde{T}$ is linear. We have also $\|Tx_n\| \leq K\|x_n\|$ and therefore

$$\|\tilde{T}x\| = \lim \|Tx_n\| \leq K \lim \|x_n\| = K\|x\|$$

If $x \in X_0$, then we can use the constant sequence (x) and hence $\tilde{T}x = Tx$. $\square$