DM534 — Øvelser Uge ??

Introduktion til Datalogi, Efterår 2021

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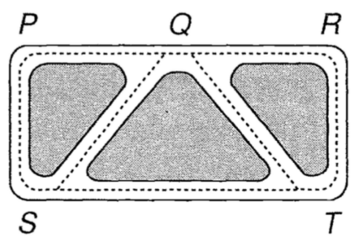
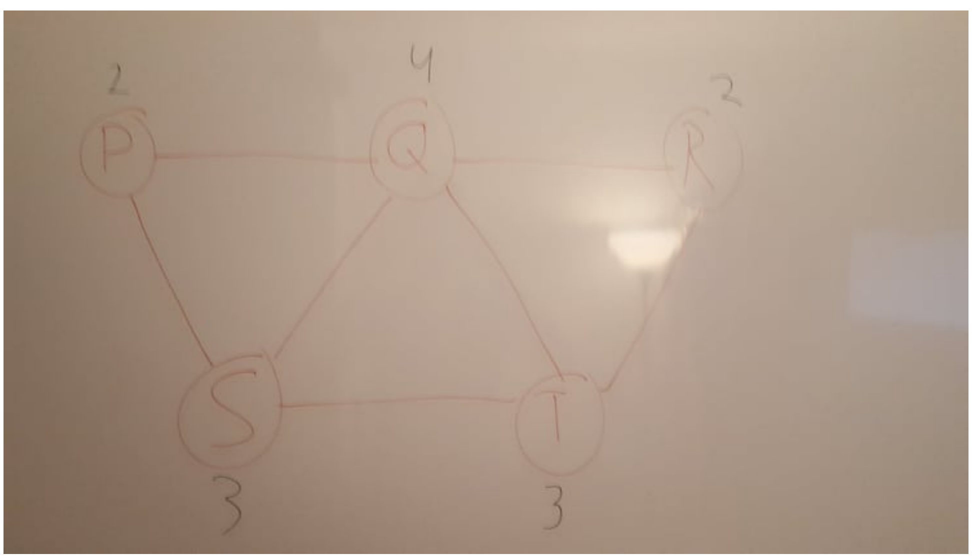
1 I

1.1

Draw the graph representing the road system in the figure below, and write down the number of vertices, the number of edges and the degree of each vertex.



SVAR:



There are 5 vertices and 7 edges. Node degrees: see numbers next to nodes on the whiteboard picure.

1.2

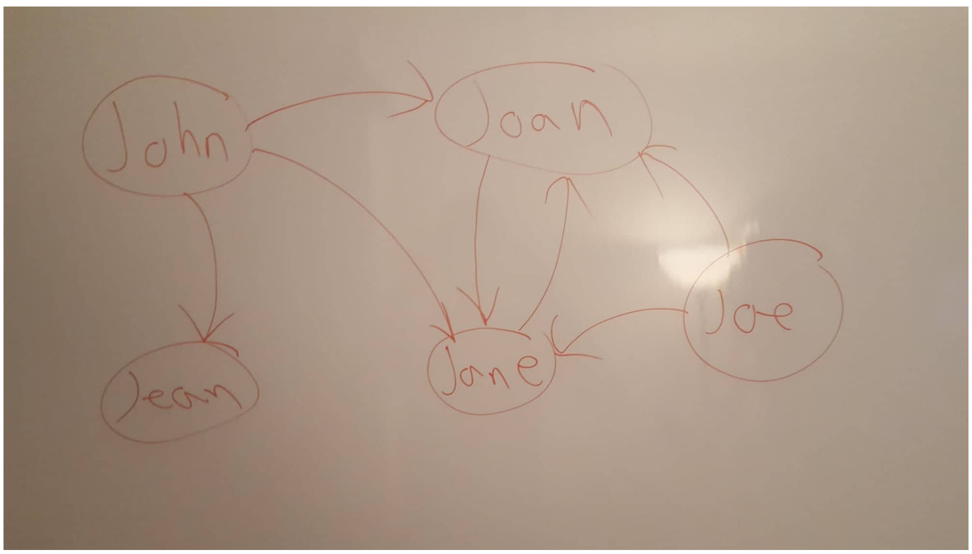
On Twitter:

(a) John follows Joan, Jean and Jane; Joe follows Jane and Joan; Jean and Joan follow each other. Draw a digraph illustrating these follow-relationships between John, Joan, Jean, Jane and Joe.

(b) Twitter has $\approx$ 313 million active users (June 2016, based on Twitter Inc.). Imagine you would like to store the digraph for the follow-relationships in an adjacency matrix that uses 4 bytes per entry on your new laptop which has 64 GB of RAM. Is this feasible?

(c) The municipality of Odense has a population of $\approx$ 200000 people. Let G be the graph where the meaning of an edge from vertex i to j is *"person i is friends with person j"*. Imagine you would like to store the adjacency matrix for this graph for the relationships in a matrix representation that uses 4 bytes per entry on your new laptop which has 64 GB of RAM. Is this feasible?

SVAR a:

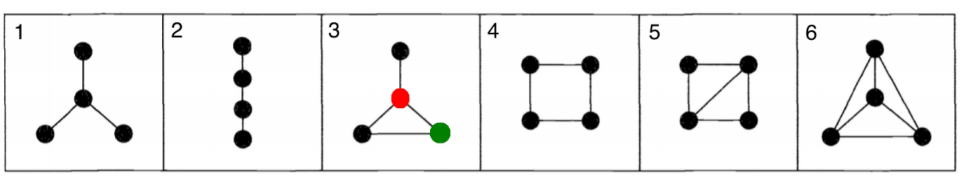


SVAR b: $(313 \cdot 10^6)^2 \cdot 4$ bytes $= 3.91876 \cdot 10^{17}$ bytes $> 64 \cdot 10^9$ bytes. No.

SVAR c: $(2 \cdot 10^5)^2 \cdot 4$ bytes $= 160 \cdot 10^9$ bytes $> 64 \cdot 10^9$ bytes. No.

1.3

Consider the following six graphs (note that the nodes do not have labels).



(a) How many walks of length 3 from the red vertex to the green vertex are there in graph 3?

(b) How many paths from the red vertex to the green vertex are there in graph 3?

(c) How many shortest paths from the red vertex to the green vertex are there in graph 3?

(d) For each of the graphs: what is the longest of all pairwise shortestpaths?

(e) Give an adjacency matrix for graph 1. Can there be different adjacency matrices for the same graph? If so, name a second adjacency matrix for graph 1. Can you find two different adjacency matrices for graph 6?

SVAR a: 4.

SVAR b: 2.

SVAR c: 1.

SVAR d: 2,3,2,2,2,1.

SVAR e:

Two different adjacency matrices for graph 1, resulting from different namings of the vertices (that is, different assignments of the vertex IDs 1, 2, 3, 4).

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

All assignments of vertex IDs result in the same adjacency matrix for graph 6.

1.4

Let A be an adjacency matrix. In the lecture you learned that the ij-entry of A^k is the number of different walks from vertex i to vertex j using exactly k edges.

(a) What is the interpretation of ij-entry of the matrix $A^1 + A^2 + A^3$?

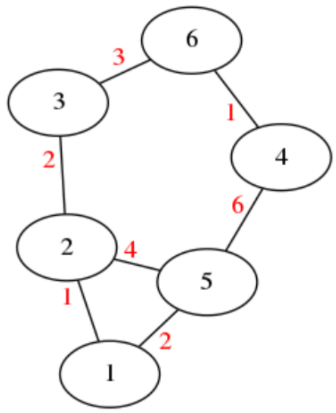
(b) Complete the following sentence with the missing expression: In a graph G with adjacency matrix A, vertex i and j $(i \neq j)$ are connected if and only if ... > 0.

SVAR a: The number of walks from i to j with length 1, 2 or 3.

SVAR b: They are connected if and only if $c_{ij} > 0$, where c_{ij} is the entry in the i'th row and j'th column in the matrix $C = A^1 + A^2 + \cdots + A^{n-1}$. It is enough to consider powers up to $k-1$, since if i og j are connected by a walk, they are also connected by a path, i.e., a walk without repetitions among the nodes (a repetition will mean that the walk contains a cycle, which can just be removed from the walk, after which it will still connect i and j). If we want the statement to also hold for $i = j$, we can just add A^0 to the sum (A^0 is defined to be the identity matrix, i.e., the matrix having ones on the diagonal and zeros elsewhere).

1.5

Let the following weighted graph (from the lecture slides, weights are depicted in red) be given:



It has the following distance matrix D:

$$D = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 3 & 7 & 2 & 6 \\ 2 & 1 & 0 & 2 & 6 & 3 & 5 \\ 3 & 3 & 2 & 0 & 4 & 5 & 3 \\ 4 & 7 & 6 & 4 & 0 & 6 & 1 \\ 5 & 2 & 3 & 5 & 6 & 0 & 7 \\ 6 & 6 & 5 & 3 & 1 & 7 & 0 \end{array}$$

(a) How many shortest path in G are of length 6? Name them. **SVAR:** 1-6, 2-4 and 4-5 (and their reversals).

(b) How long is the longest of all pairwise shortest paths in the graph? Are there several longest shortest paths? **SVAR:** 1-4 and 5-6 (and their reversals) are 7 long.

(c) How many paths in G are of length 6? (Note: a path does not necessarily need to be a shortest path.) Name them. **SVAR:** 1-2, 1-6, 2-4, 3-5, 4-5 (and their reversals).

1.6

Assume in this exercise that all weights on edges are non-negative values.

(a) In a graph G with $n = 6$ vertices, how many matrix-matrix multiplication operations are needed in the worst case in order to compute the distance matrix D, when the method of repeated squaring is used to compute D? **SVAR:** $\lceil \log_2(6) \rceil = 3$

(b) In a graph G with $n = 200$ vertices, how many matrix-matrix multiplications are needed in the worst case in order to compute the distance matrix D, when the method of repeated squaring is used to compute D? **SVAR:** $\lceil \log_2(200) \rceil = 8$

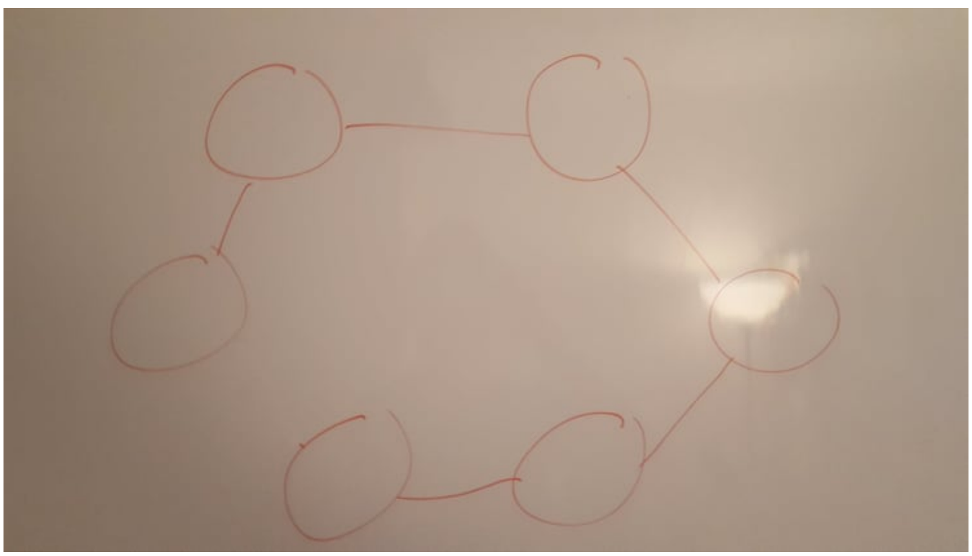
(c) Can you find a graph G with $n = 6$ vertices, for which $W^4 \neq W^5$? If so, depict it.

(d) Can you find a graph G with $n = 6$ vertices, for which $W^5 \neq W^6$? If so, depict it. **SVAR:** No, all powers W^i for $i \geq n-1$ will be the same matrix. This follows from the theorem on the slide containing the page number 41, combined with the fact that paths with more than $n-1$ edges cannot contribue new shortest paths, as is argued on the slide containing the page number 49.

(e) Can you find a graph G with $n = 6$ vertices, for which $W^1 = W^2$? If so, depict it.

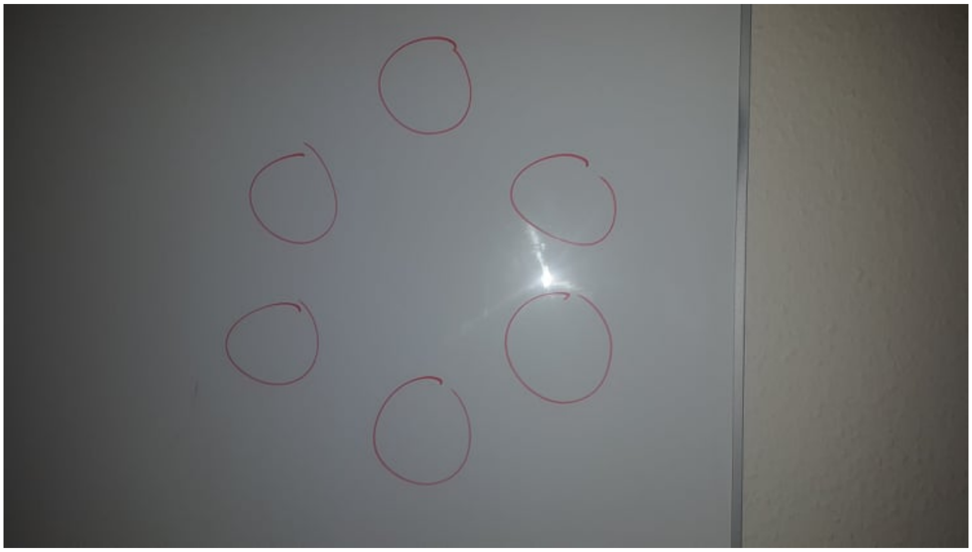
(f) What is the computational runtime in order to compute the distance matrix D for a graph G with n vertices if the method of repeated squaring is used to compute D? **SVAR:** $n^3 \log_2(n)$.

SVAR c:



(Edges all have weight one.)

SVAR e:

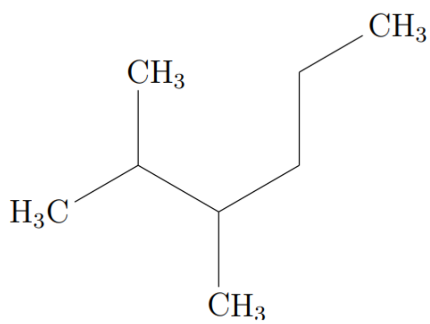


Another option is the unordered graph having an edge (of weight one) between *each* pair of nodes.

2 II

2.1

Consider the following molecule (it's called 2,3-Dimethylhexane, see `https://en.wikipedia.org/wiki/2,3-Dimethylhexane`):



(a) How many carbon atoms does this molecule have?

(b) Draw the graph G corresponding to the carbon backbone of the molecule.

(c) Give the edge weight matrix W for the graph G.

(d) Use your brain or the Java program ShortestPaths.java to infer the distance matrix (Hint: the graph is rather simple, you won't need a program for that.)

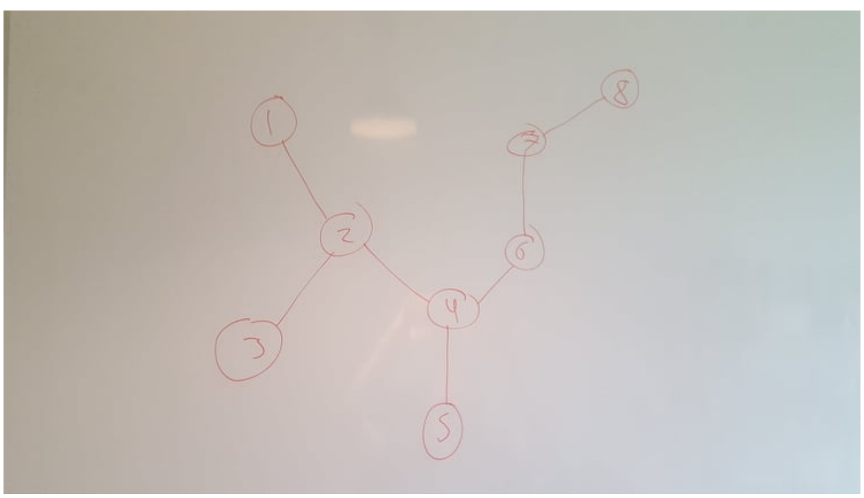
(e) What is the Wiener Index $\mathcal{W}(G)$?

(f) How many shortest paths of length 3 $i \to ... \to j$ with $i < j$ are in G?

(g) Using Wiener's method for predicting the boiling point, what is your prediction for 2,3-Dimethylhexane?

SVAR a: 8.

SVAR b:



SVAR c:

$$\begin{pmatrix} 0 & 1 & \infty & \infty & \infty & \infty & \infty & \infty \\ 1 & 0 & 1 & 1 & \infty & \infty & \infty & \infty \\ \infty & 1 & 0 & \infty & \infty & \infty & \infty & \infty \\ \infty & 1 & \infty & 0 & 1 & 1 & \infty & \infty \\ \infty & \infty & \infty & 1 & 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & 1 & 0 & 1 \\ \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

SVAR d:

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 & 5 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 & 4 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 & 2 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 & 1 \\ 5 & 4 & 5 & 3 & 4 & 2 & 1 & 0 \end{pmatrix}$$

SVAR e:

(1+2+2+3+3+4+5)+(1+1+2+2+3+4)+(2+3+3+4+5)+(1+1+2+3)+(2+3+4)+(1+2)+1

$$= 20 + 13 + 17 + 7 + 9 + 3 + 1 = 70$$

SVAR f: 1-5, 1-6, 2-7, 3-5, 3-6, 4-8, 5-7. Total of 7.

SVAR g:

$$t_B = t_0 - \left(\frac{98}{n^2}(w_0 - \mathcal{W}(G)) + 5.5 \cdot (p_0 - p) \right)$$

$$t_0 = 745.42 \cdot \log_{10}(n + 4.4) - 689.4 = 745.42 \cdot \log_{10}(8 + 4.4) - 689.4 = 125.658$$

$$w_0 = \frac{1}{6} \cdot (n+1) \cdot n \cdot (n-1) = \frac{1}{6} \cdot (8+1) \cdot 8 \cdot (8-1) = 84$$

$$p_0 = n - 3 = 8 - 3 = 5$$

$$p = 7$$

$$t_B = 125.658 - \left(\frac{98}{8^2}(84 - 70) + 5.5 \cdot (5 - 7) \right) = 115.2205$$