Last time:

P: poly-time solvable
NP: poly-time checkable certificates
NP-hard: „at least as hard" as any NPC-problem
NPC: If *any* problem in NPC is in P, they *all* are.

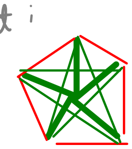
TSP: complete graph, $\underbrace{\text{symm. weights},\ w_{ii}=0,\ w\geq 0}$
Metric weights: + Δ-ineq.

TSP inapproximable (Thm 2.9)
Metric TSP:

Nearest Addition
Double Tree

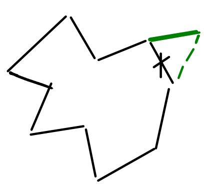
} 2-approx.

Tight:

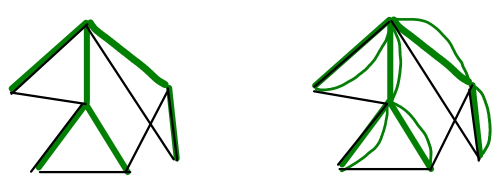


MST OPT DT

NA:

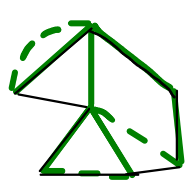


DT:



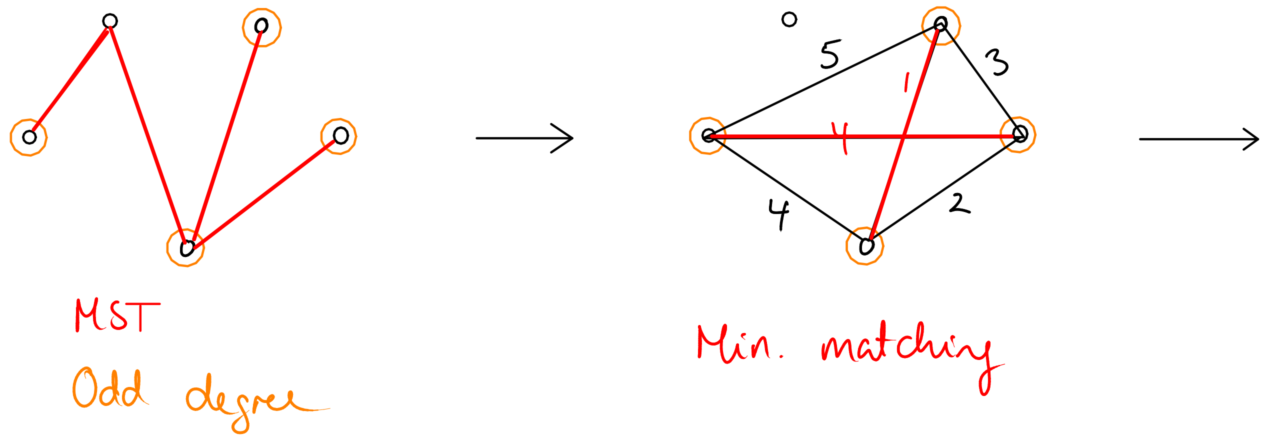
$\boxed{\begin{array}{l} C_{DT} \leq 2\cdot C(MST) \\ C(MST) \leq C_{OPT} \end{array}}$ by Δ-ineq. since a ST can be created by deleting an edge from OPT.

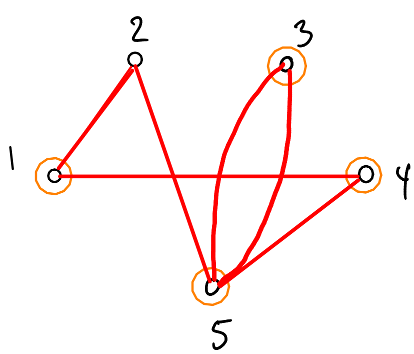
Christofide's Alg:



Christofide's Algorithm

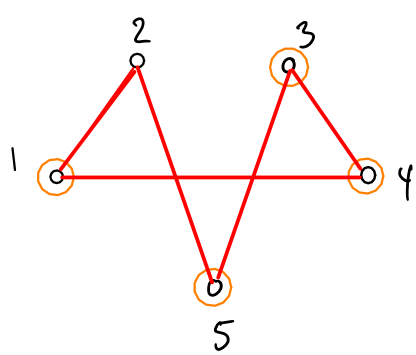
Next idea: Not necessary to add n-1 edges to obtain even degree for all vertices.
Instead: add a minimum perfect matching on vertices of odd degree in the MST.





Euler tour: ⟨1, 2, 5, 3, 5, 4, 1⟩

↓ short cutting



TSP tour: ⟨1, 2, 5, 3, 4, 1⟩

Note that it is always possible to find a perfect matching, since there is always an even # odd degree vertices in T.

Christofide's Algorithm (CA)

$T \leftarrow$ MST

$M \leftarrow$ minimum perfect matching on odd degree vertices in T

ETour $\leftarrow$ Euler tour in the subgraph $(V, E(T) \cup M)$

Tour $\leftarrow$ vertices in order of first appearance in ETour

Theorem 2.13

Christofide's Algorithm is a $3/2$-approx. alg.

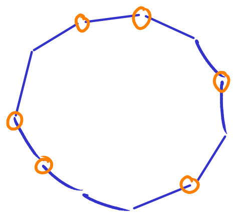
Proof:

By the triangle inequality,

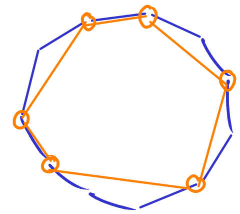
$c_{CA} \leq c(T) + c(M)$, where

$c(T) \leq c_{OPT}$, by the arguments above, and

$c(M) \leq \frac{1}{2} c_{OPT}$, by the arguments below.



short cutting $\longrightarrow$



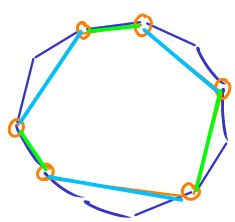
Optimal TSP tour

Odd degree vertices in T

C: cost of orange cycle

$C \leq c_{OPT}$, by Δ-ineq.

Since the cycle on the odd degree vertices has an even #edges, it consists of two perfect matchings:



$$C = C + C$$

$$\Downarrow$$

$$\min\{C, C\} \leq \tfrac{1}{2} \cdot C \leq \tfrac{1}{2} \cdot c_{OPT}$$

Since M is a <u>minimum</u> matching on the odd degree vertices,

$$c(M) \leq \min\{C, C\} \leq \tfrac{1}{2} \cdot c_{OPT}$$

$\square$

No alg. with an approx. ratio better than $3/2$ is currently known. Moreover:

Theorem 2.14

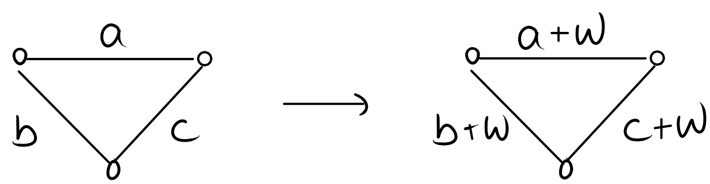
For $\alpha < \frac{220}{219}$, $\nexists$ α-approx. alg. for Metric TSP

The result of Thm 2.14 is from 2000.
In 2015, the same result was proven for $\alpha < \frac{185}{184}$.

Sheet 1

1. a) Add $W = \max_{e \in E} \{w_e\}$ to all edge weights.

The resulting weights are metric:



$$a+W \leq 2W \leq (b+W)+(c+W)$$

b) For any tour T

$\underline{T}$ optimal in $G \iff T$ optimal in G'

For any tour T in G, let $w(T)$ be the total weight of T in G and let $w'(T)$ be the total weight of T with the modified weights.

Then $$w'(T) = w(T) + \underbrace{nW}_{\text{This part is the same for any tour}}$$

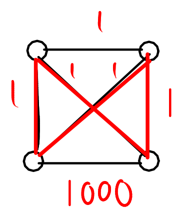
only this part can vary

Hence, w' is minimized when w is minimized

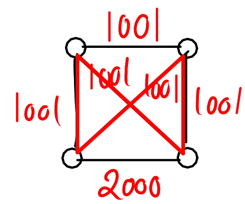
c) Contradiction with inapproximability?

The reduction to the metric case is not approx. factor preserving.

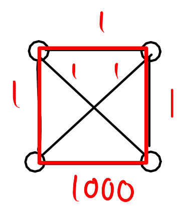
<u>Ex</u>:



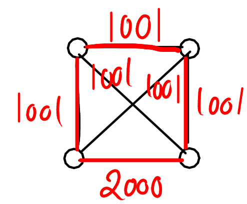
$W = 4$



$W = 4004$



$W = 1003$



$W = 5003$

ratio ≈ 250

ratio $\approx 5/4$

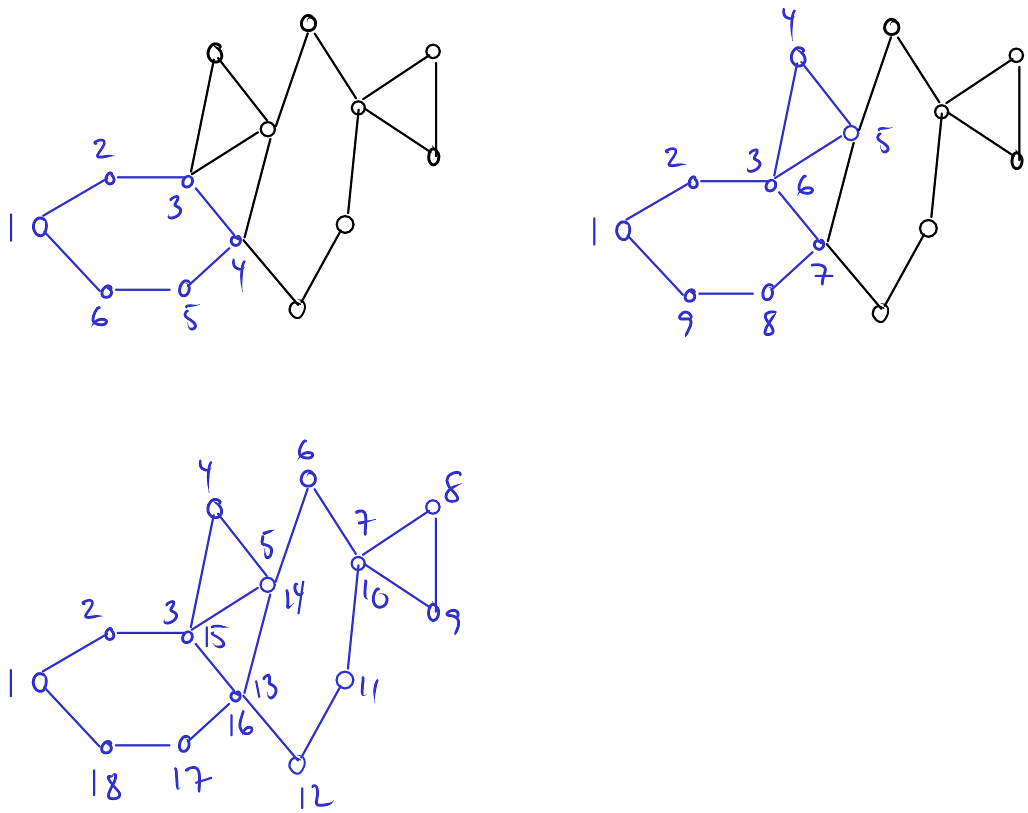
② Argue that <u>metric</u> TSP is NP-hard.

The graph used in the reduction just before Thm 2.9 in the lecture notes is metric.

③ Alg. for Euler tour in connected graph

$v \leftarrow$ any vertex in the graph
Follow non-traversed edges, starting in v, until reaching a vertex with no non-traversed edges
While $\exists$ non-traversed edges
    $v \leftarrow$ vertex with both traversed and non-traversed edges
    Follow non-traversed edges, starting in v, until reaching a vertex with no non-traversed edges

Ex:



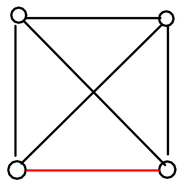
Correctness:

When reaching a vertex with no non-traversed edges, the vertex has an even #traversed edges. This can only be v, so we have produced a tour.

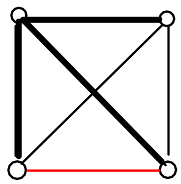
Since the graph is connected, there must be a non-traversed edge leaving the tour, if there are still non-traversed edges.

4. Christofide's vs Double Tree

a) Example where C. does better than D.T.

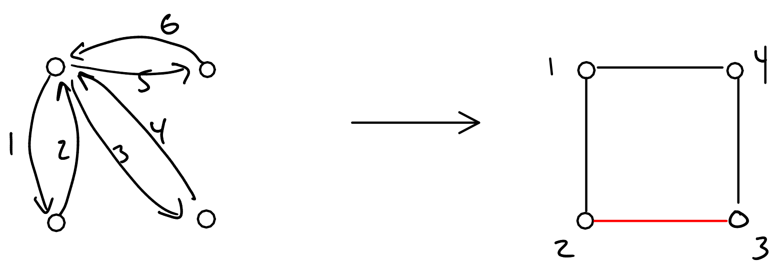


Black edges have weight 1.
Red edge has weight 2.

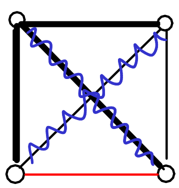


Thick edges: MST

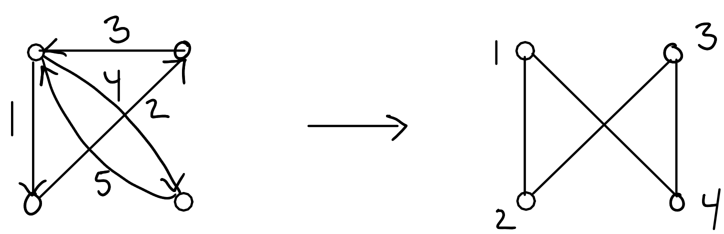
Double Tree:



Christofide's:

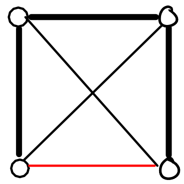


: minimum matching

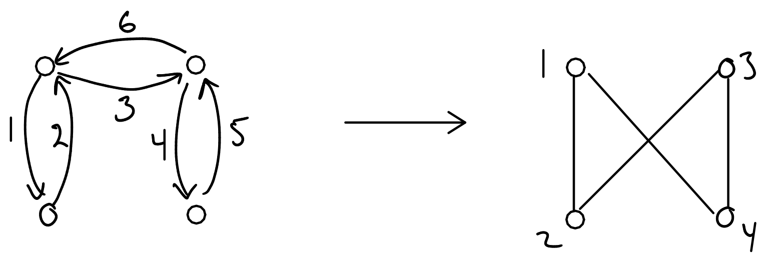


b) Example where D.T. does better than C.

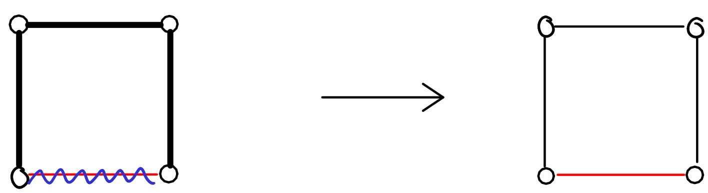
Same graph, different MST:



Double Tree:



Christofide's:



c) How many nodes are needed?

4 suffice

3 are two few, since that gives only one possible tour.