Chapter 5: Maximum Satisfiability

SAT: For a given boolean formula φ in CNF, does there exist a truth assignment satisfying φ?

Conjunctive normal form (CNF): the formula is a conjunction ($\wedge$) of disjunctions ($\vee$). Each disjunction is called a clause.

Ex: $$\varphi \equiv \overbrace{(x_1 \vee \bar{x}_2 \vee x_3)}^{\text{clause } C_1} \wedge \overbrace{\bar{x}_3}^{C_2} \wedge \overbrace{(x_1 \vee x_2)}^{C_3}$$

positive literal

negative literal

x_1, x_2, x_3 are variables

C_j has length/size ℓ_j:

$\ell_1 = 3$, $\ell_2 = 1$, $\ell_3 = 2$

$x_1 \leftarrow T$, $x_3 \leftarrow F$ will satisfy φ

MAX SAT

Input: Boolean formula φ in CNF
with variables $x_1, x_2, \dots, x_n$
and clauses $C_1, C_2, \dots, C_m$
Each clause, C_j, has a weight w_j

Output: Truth assignment maximizing the
total weight of satisfied clauses

Ex: $(x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$w_1 = 2 \qquad w_2 = 2 \qquad w_3 = 1 \qquad w_4 = 3$

$x_1 \leftarrow T$, $x_2 \leftarrow F$, $x_3 \leftarrow T$ satisfies C_1, C_2, C_4
with a total weight of 7.

This is optimal, since we cannot satisfy
all clauses:

C_2 requires $x_3 \leftarrow T$
C_3 then requires $x_2 \leftarrow T$
C_1 then requires $x_1 \leftarrow T$
But then C_4 is false.

SAT, and hence, MAX SAT, is NP-hard.
How can we approximate?

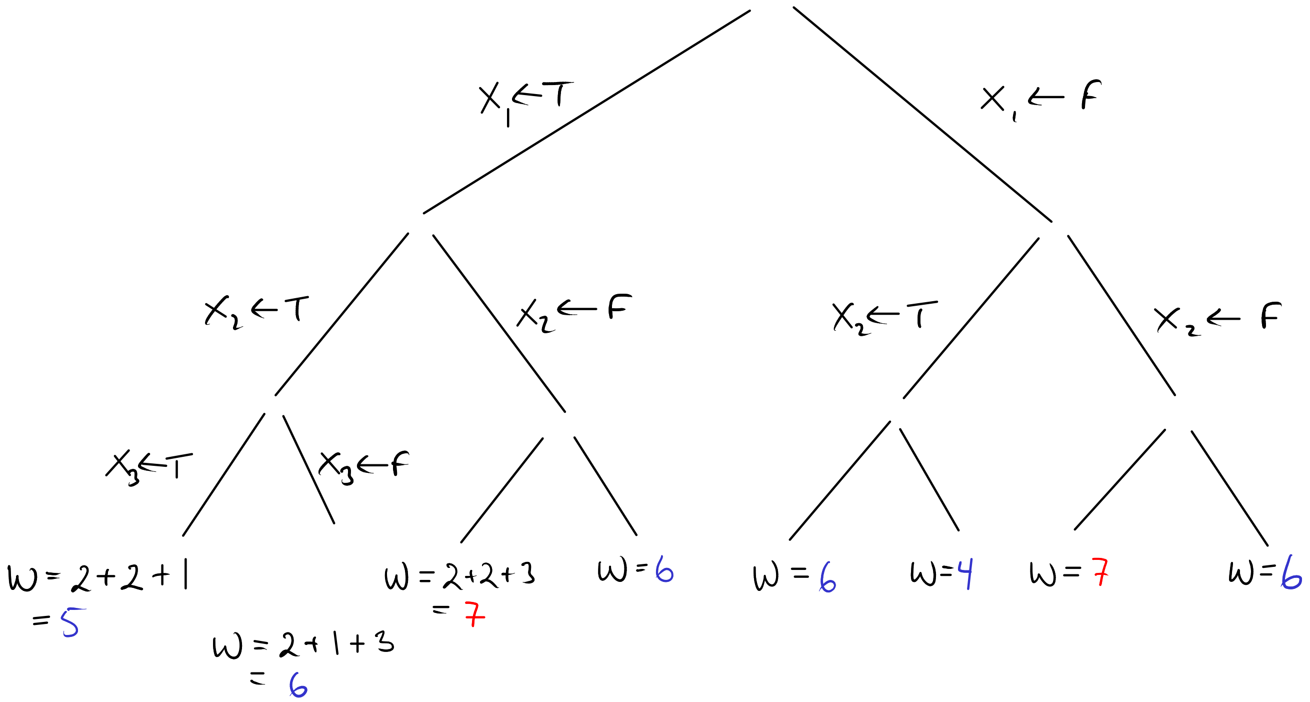
Section 5.1: A simple randomized alg.

Consider the following alg:

```
Rand
For i←1 to n
    With prob. $1/2$ set $x_i$ true
```

This corresponds to choosing a solution uniformly at random.

Ex: $(x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$



Thus, for this example,

$$E[\text{Rand}] = \frac{1}{8}(5+6+7+6+6+4+7+6) = 5\tfrac{7}{8}$$

We don't need to calculate the weight of each possible output...

Instead, we can calculate the exp. weight of each clause:

$$(x_1 \vee \overline{x}_2) \wedge x_3 \wedge (x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$$

↑ satisfied, unless $x_1 \equiv F$ and $x_2 \equiv T$, i.e., satisfied with prob. $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

↖ satisfied w. prob. $1 - \frac{1}{2} = \frac{1}{2}$

↑ satisfied w. prob. $1 - 1/2 \cdot 1/2 \cdot 1/2 = 7/8$

Thus,

$$E[\text{Rand}] = 3/4 \cdot 2 + 1/2 \cdot 2 + 3/4 \cdot 1 + 7/8 \cdot 3 = 5\,7/8$$

In general, clause C_j is satisfied with prob. $1 - (\frac{1}{2})^{l_j}$.

We let $W = \sum_{j=1}^{m} w_j$.

Theorem 5.1: Rand is a $\frac{1}{2}$-approx. alg

Proof:

$OPT \leq W$

By linearity of expectation:

$$E[Rand] = \sum_{j=1}^{m} \left(1-\left(\tfrac{1}{2}\right)^{l_j}\right) w_j$$

$$\geq \tfrac{1}{2} W \qquad , \quad \text{since } l_j \geq 1$$

□

In Section 5.1 we got a simple algorithm with a guarantee on the expected performance. We can turn it into a guarantee on the worst-case performance:

Section 5.2: Derandomization

Ex from before:

$$\phi : (x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

$$w_1 = 2 \qquad w_2 = 2 \qquad w_3 = 1 \qquad w_4 = 3$$

If we let $\boxed{x_1 \leftarrow T}$, the formula becomes

$$\phi_T : T \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$$

and

$$E[Rand(\phi_T)] = 2 + \tfrac{1}{2} \cdot 2 + \tfrac{3}{4} \cdot 1 + \tfrac{3}{4} \cdot 3 = \textcircled{6}$$

Or, recalling the probability tree,

$$E[Rand(\phi_T)] = \tfrac{1}{4}(5+6+7+6) = 6$$

Similarly, if we let $x_1 \leftarrow F$, the formula becomes

$$\phi_F : \bar{x}_2 \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge T$$

and

$$E[Rand(\phi_F)] = \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 2 + \tfrac{3}{4} \cdot 1 + 3 = 5\tfrac{3}{4}$$

Or $E[Rand(\phi_F)] = \frac{1}{4}(6+4+7+6) = 5\frac{3}{4}$

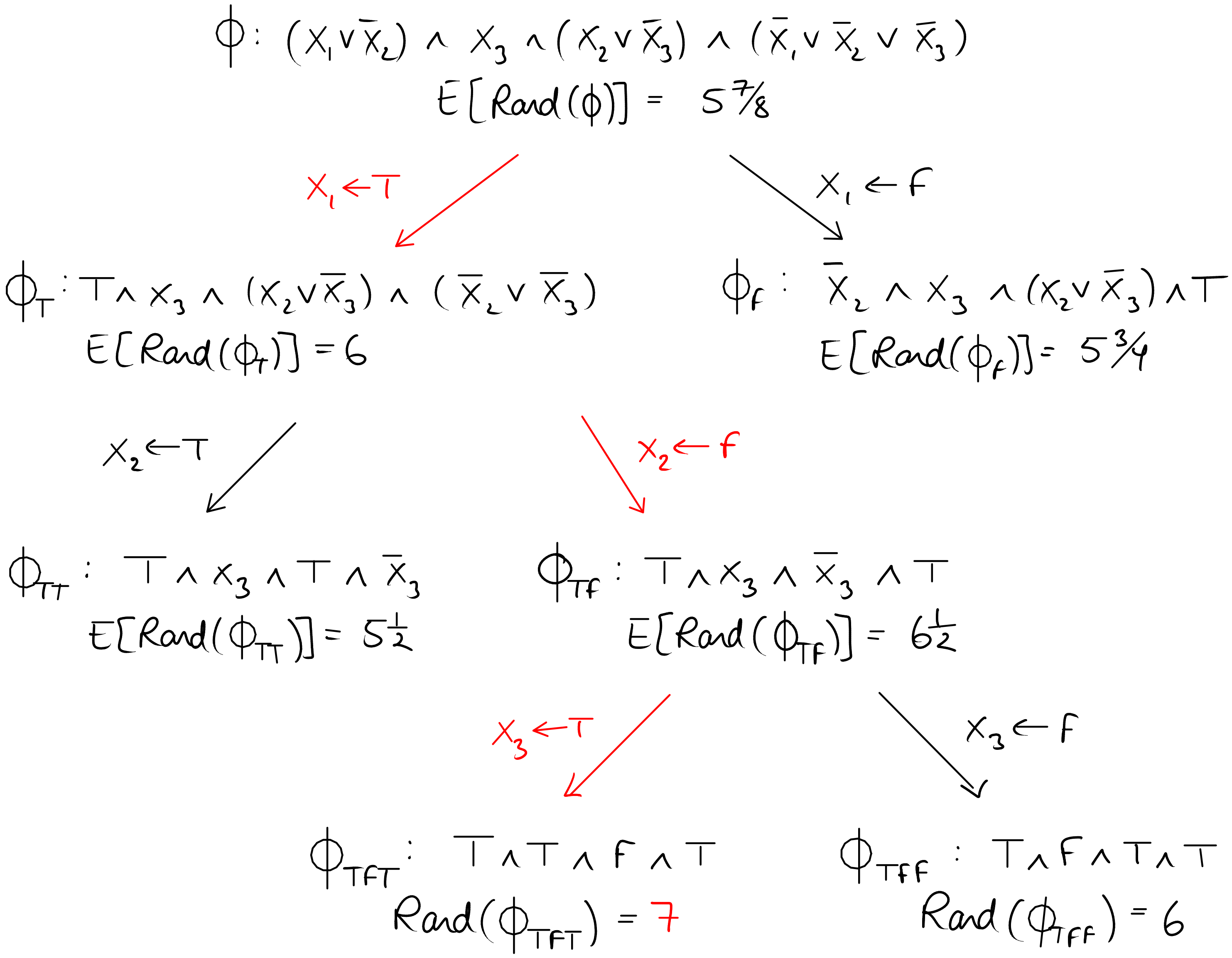
Note that $E[Rand]$ is the average of 6 and $5\frac{3}{4}$:

$$E[Rand] = \tfrac{1}{2} \cdot E[Rand(\phi_T)] + \tfrac{1}{2} \cdot E[Rand(\phi_F)]$$

Thus,

$$\max\{E[Rand(\phi_T)], E[Rand(\phi_F)]\} \geq E[Rand]$$

I.e., one of the leaves in the left part of the probability tree must have an exp. value of ≥ 6.


$\phi : (x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$
$E[Rand(\phi)] = 5\,^7/_8$
$x_1 \leftarrow T$
$x_1 \leftarrow F$
$\phi_T : T \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$
$E[Rand(\phi_T)] = 6$
$\phi_F : \bar{x}_2 \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge T$
$E[Rand(\phi_F)] = 5\,^3/_4$
$x_2 \leftarrow T$
$x_2 \leftarrow F$
$\phi_{TT} : T \wedge x_3 \wedge T \wedge \bar{x}_3$
$E[Rand(\phi_{TT})] = 5\frac{1}{2}$
$\phi_{TF} : T \wedge x_3 \wedge \bar{x}_3 \wedge T$
$E[Rand(\phi_{TF})] = 6\frac{1}{2}$
$x_3 \leftarrow T$
$x_3 \leftarrow F$
$\phi_{TFT} : T \wedge T \wedge F \wedge T$
$Rand(\phi_{TFT}) = 7$
$\phi_{TFF} : T \wedge F \wedge T \wedge T$
$Rand(\phi_{TFF}) = 6$

In general:

$$\max\{E[Rand(\phi_T)], E[Rand(\phi_F)]\} \geqslant E[Rand] \geqslant \tfrac{1}{2}W,$$

The same is true for ϕ_T and ϕ_F:

$$\max\{E[Rand(\phi_{TT})], E[Rand(\phi_{TF})]\} \geqslant E[Rand(\phi_T)]$$

and

$$\max\{E[Rand(\phi_{FT})], E[Rand(\phi_{FF})]\} \geqslant E[Rand(\phi_F)]$$

Inductively, this proves that the following alg. is a $\frac{1}{2}$-approx. alg.:

```
DeRand($\phi$)
For $i \leftarrow 1$ to $n$
    If $E[Rand(\phi_{x_1 \cdots x_{i-1} T})] \geqslant E[Rand(\phi_{x_1 \cdots x_{i-1} F})]$
        $x_i \leftarrow T$
    Else
        $x_i \leftarrow F$
```

This method of derandomization is sometimes called the method of conditional expectations. (We calculate the conditional exp. of Rand given that $x_i \leftarrow T$ and given that $x_i \leftarrow F$.)

Note that short clauses are „harder" than long clauses:

If all clauses have $\ell \geq 2$, (De)Rand is a $\frac{3}{4}$-approx. alg.

(In Section 5.3, we will pursue the obs. to obtain a ≈ 0.6-approx. alg.)

If all clauses have $\ell \geq 3$, (De)Rand is a $\frac{7}{8}$-approx alg.

In some sense, this is optimal:

MAX E3SAT: The special case of MAX SAT where $\ell = 3$ for all clauses.

Theorem 5.2:

$\exists \varepsilon > 0 : \exists \left(\frac{7}{8}+\varepsilon\right)$-approx alg for MAX E3SAT $\Rightarrow$ $P = NP$

Section 5.3: A biased rand. alg.

Since unit clauses (clauses of exactly one literal) are the „hardest", we should focus on these to obtain a better approx. ratio.

For each i, $1 \le i \le n$, we define:

$$u_i = \begin{cases} \text{weight of unit clause } x_i, & \text{if it exists} \\ 0, & \text{otherwise} \end{cases}$$

$$v_i = \begin{cases} \text{weight of unit clause } \bar{x}_i, & \text{if it exists} \\ 0, & \text{otherwise} \end{cases}$$

Idea: If $u_i \ge v_i$, set x_i true with prob. $> \frac{1}{2}$, and vice versa.

For ease of presentation, assume that

$$u_i \ge v_i, \quad 1 \le i \le n$$

Why is this not a restriction?

Thus, each variable will be set true with prob. $> \frac{1}{2}$:

For any $p > \frac{1}{2}$, we define the following alg:

Rand$_p$

```
For $i \leftarrow 1$ to $n$
    With prob. $p$ set $x_i$ true
```

What is an optimal value of p?

Lemma 5.4

For any clause C_j which does not consist of one negated variable, Rand$_p$ satisfies C_j with prob $\geq \min\{p, 1-p^2\}$

Proof:

If $l_j = 1$, C_j consists of one unnegated variable. In this case, C_j is satisfied with prob. p.

If $l_j = 2$, the worst case is if both literals are negated variables, since $p > \frac{1}{2}$. Thus, in this case, C_j is satisfied with prob. $\geq 1-p^2$.

If $l_j \geq 3$, the prob. of C_j being satisfied is at least the worst-case prob. for $l_j = 2$. □

Lemma 5.6: $OPT \leq W - \sum_{i=1}^{n} v_i$

Proof:

By assumption, $u_i \geq v_i$, for all i.
Thus, if $v_i > 0$, there is both an x_i- and an $\bar{x}_i$-clause. Both clauses cannot be satisfied.
Thus, for each $v_i > 0$, there is an unsatisfied clause of weight $\geq \min\{u_i, v_i\} = v_i$. $\square$

We can obtain an alg. with approx. ratio $\frac{1}{2}(\sqrt{5}-1) \approx 0{,}618$:

Theorem 5.7:

For $p = \frac{1}{2}(\sqrt{5}-1)$, Rand$p$ is a p-approx. alg.

Proof:

By Lemma 5.4,

$$E[\text{Rand}_p] \geq \min\{p, 1-p^2\} \cdot \overbrace{\left(W - \sum_{i=1}^{n} v_i\right)}^{\text{Total weight of clauses that are not negated unit clauses}}$$

$$= p\left(W - \sum_{i=1}^{n} v_i\right), \text{ for } p = \tfrac{1}{2}(\sqrt{5}-1) :$$

$$1 - \left(\tfrac{1}{2}(\sqrt{5}-1)\right)^2 = 1 - \tfrac{1}{4}(5+1-2\sqrt{5}) = 1 - 3/2 + \tfrac{1}{2}\sqrt{5}$$

$$= \tfrac{1}{2}(\sqrt{5}-1)$$

By Lemma 5.6, $\text{OPT} \leq W - \sum_{i=1}^{n} v_i$.

Hence, for $p = \tfrac{1}{2}(\sqrt{5}-1)$, $\dfrac{E[\text{Rand}_p]}{\text{OPT}} \geq p$ $\square$

Note that

$\boxed{\text{Rand}_p \text{ can be derandomized exactly like Rand}}$