Section 5.5 : Choosing the better of two solutions

Combining the alg.s of Sections 5.1 and 5.4 gives a better approx. factor than using any one of them separately. This is because they have different worst-case inputs:

Rand satisfies clause C_j with prob. $p_R = 1 - \left(\frac{1}{2}\right)^{l_j}$.

RandRounding satisfies C_j with prob. $p_{RR} \geq \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^*$.

While p_R increases with l_j, the lower bound on p_{RR} decreases with l_j.

BestOfTwo(ϕ)

$\vec{x}_R \leftarrow \text{Rand}(\phi)$
$\vec{x}_{RR} \leftarrow \text{RandRounding}(\phi)$
If $w(\phi, \vec{x}_R) \geq w(\phi, \vec{x}_{RR})$
    Return $\vec{x}_R$
Else
    Return $\vec{x}_{RR}$

Note that

> BestOfTwo is derandomized by using the derandomized versions of Rand and RandRounding.

Theorem 5.11 : BestOfTwo is a 3/4-approx. alg.

Proof:

$$E[\text{BestOfTwo}(\phi)] = E\left[\max\{\text{Rand}(\phi),\ \text{RandRounding}(\phi)\}\right]$$

$$\geq E\left[\tfrac{1}{2}\text{Rand}(\phi) + \tfrac{1}{2}\text{RandRounding}(\phi)\right]$$

$$= \tfrac{1}{2}E[\text{Rand}(\phi)] + \tfrac{1}{2}E[\text{RandRounding}(\phi)], \text{ by lin. of exp.}$$

$$\geq \frac{1}{2}\sum_{j=1}^{m}\left(1-2^{-l_j}\right)w_j + \frac{1}{2}\sum_{j=1}^{m}\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)z_j^* w_j$$

$$\geq \sum_{j=1}^{m} z_j^* w_j \cdot \underbrace{\frac{1}{2}\left(1-2^{-l_j} + 1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)}_{\equiv\, p_j}, \text{ since } z_j^* \leq 1.$$

For $l_j = 1$, $p_j = \frac{1}{2}\left(1-\frac{1}{2}+1-0\right) = \frac{3}{4}$

For $l_j = 2$, $p_j = \frac{1}{2}\left(1-\frac{1}{4}+1-\left(1-\frac{1}{2}\right)^2\right) = \frac{3}{4}$

For $l_j \geq 3$, $p_j \geq \frac{1}{2}\left(1-\frac{1}{8}+1-\frac{1}{e}\right) > \frac{3}{4}$

Hence,

$$E[\text{BestOfTwo}] \geq \sum_{j=1}^{m} z_j^* w_j \cdot \frac{3}{4} \geq \frac{3}{4}\cdot \text{OPT}$$

□

Section 5.6: Nonlinear randomized rounding

RandRounding$_f(\phi)$

$(\vec{y^*}, \vec{z^*}) \leftarrow$ opt. sol. to LP_ϕ
For $i \leftarrow 1$ to n
  Set x_i true with prob. $f(y_i^*)$

Theorem 5.12

RandRounding$_f$ is a $3/4$-approx. alg., if $1 - 4^{-x} \leq f(x) \leq 4^{x-1}$

Proof:

Prob. that C_j is not satisfied:

$$\bar{p}_j = \prod_{i \in P_j} (1 - f(y_i^*)) \prod_{i \in N_j} f(y_i^*)$$
$$\leq \prod_{i \in P_j} 4^{-y_i^*} \prod_{i \in N_j} 4^{y_i^* - 1}$$
$$= 4^{-\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} 1 - y_i^*\right)}$$
$$\leq 4^{-z_j^*}$$

Prob. that C_j is satisfied:

$$p_j = 1 - \bar{p}_j \geq 1 - 4^{-z_j^*}$$
$$\geq 0 + \left(\frac{3}{4} - 0\right) z_j^*, \quad \text{by Fact 5.9}$$
$$= \frac{3}{4} z_j^*$$

$\square$

Ex: $\phi \equiv (x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2})$

$w_1 = w_2 = w_3 = w_4 = 1$

OPT = 3

$y_1 = y_2 = \frac{1}{2} \Rightarrow Z = 4$

Hence, the integrality gap for the IP problem for MaxSat is

$$\min_{\psi} \left\{ \frac{Z^*_{IP_\psi}}{Z^*_{LP_\psi}} \right\} \leq \frac{Z^*_{IP_\phi}}{Z^*_{LP_\phi}} = \frac{3}{4}$$

On the other hand, the proof that $Rand_f$ is a 3/4-approx. alg. shows that for any instance ψ of MaxSat, $Rand_f(\psi) \geq \frac{3}{4} Z^*_{LP_\psi}$. Hence,

$$\frac{Z^*_{IP_\psi}}{Z^*_{LP_\psi}} \geq \frac{Rand_f(\psi)}{Z^*_{LP_\psi}} \geq \frac{3}{4}$$

Hence, the integrality gap is exactly $\frac{3}{4}$.

The upper bound of $\frac{3}{4}$ on the integrality gap shows that we cannot prove an approx. factor better than $\frac{3}{4}$, if the approximation guarantee is based on a comparison to Z^*_{LP}:

$$\min_{\psi} \left\{ \frac{ALG(\psi)}{Z^*_{LP_\psi}} \right\} \leq \min \left\{ \frac{OPT(\psi)}{Z^*_{LP_\psi}} \right\} \leq \frac{3}{4}$$