Techniques: (with Set Cover as an example)

- Solve LP and round solution (Sec. 1.3 + 1.7)
- Primal-dual alg.: combinatorial alg. based on LP formulation (Sec. 1.4 + 1.5)
- Greedy alg. (Sec. 1.6)

Section 1.2: Set Cover as an LP

Set Cover

Input:

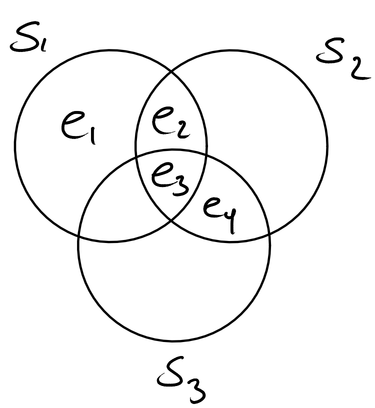
$E = \{e_1, e_2, \dots, e_n\}$

$\mathcal{S} = \{S_1, S_2, \dots, S_m\}$, where $S_j \subseteq E$ has weight w_j.

Objective: Find a cheapest possible subset of $\mathcal{S}$ covering all elements

OPT: value (total weight) of optimum solution

Ex:



$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

$\{S_1, S_2\}$ is a sol. of total weight 3.

This is optimal, so OPT $= 3$ for this instance of Set Cover.

To cover e_1, we need S_1

— " — e_2 — " — S_1 or S_2

— " — e_3 — " — S_1, S_2 or S_3

— " — e_4 — " — S_2 or S_3

IP-formulation:

$$\min \quad x_1 w_1 + x_2 w_2 + x_3 w_3$$

$$\begin{aligned} \text{s.t.} \quad & x_1 \geq 1 \\ & x_1 + x_2 \geq 1 \\ & x_1 + x_2 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \in \{0,1\} \end{aligned}$$

More generally:

IP for Set Cover

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \sum_{j:\, e_i \in S_j} x_j \geq 1 , \qquad i = 1, 2, \dots, n$$

$$x_j \in \{0, 1\} , \qquad j = 1, 2, \dots, m$$

Z^*_{IP} : optimum solution value, i.e., $Z^*_{IP} = OPT$

LP-relaxation

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \sum_{j:\, e_i \in S_j} x_j \geq 1 , \qquad i = 1, 2, \dots, n$$

$$0 \leq x_j \leq 1 , \qquad j = 1, 2, \dots, m$$

redundant

Z^*_{LP} : Optimum solution value

Note that $\boxed{Z^*_{LP} \leq Z^*_{IP} = OPT}$

Section 1.3: A deterministic rounding algo.

The frequency of an element e is the #sets containing e:

$$f_e = |\{S \in \mathcal{S} \mid e \in S\}|$$

The frequency of an instance of Set Cover:

$$f = \max_{e \in E} \{f_e\}$$

Alg. 1 for Set Cover: LP-rounding

Solve LP

$I \leftarrow \{ j \mid x_j \geq \frac{1}{f} \}$

We prove that Alg 1 produces a set cover (Lemma 1.5) of total weight $\leq f \cdot OPT$ (Thm 1.6)

Lemma 1.5

$\boxed{\{S_j \mid j \in I\} \text{ is a set cover}}$

Proof:

For each $e_i \in E$, $\sum_{j: e_i \in S_j} x_j \geq 1$.

Since $\sum_{j: e_i \in S_j} x_j$ has at most f terms, at least one of the terms is at least $\frac{1}{f}$.

Thus, there is a set S_j s.t. $e_i \in S_j$ and $x_j \geq \frac{1}{f}$.

This j is included in I □

Thm 1.6

Alg. 1 is an f-approx. algo. for Set Cover.

Proof:

Correct by Lemma 1.5

Poly, since LP-solving is poly.

Approx. factor f:

Each x_j is rounded up to 1, only if it is already at least $\frac{1}{f}$.

Thus, each x_j is multiplied by at most f, i.e.,

$$\sum_{j \in I} w_j \leq \sum_{j \in I} f \cdot x_j \cdot w_j \leq \sum_{j=1}^{m} f \cdot x_j \cdot w_j = f \cdot Z_{LP}^* \leq f \cdot OPT$$

□

The Vertex Cover problem is a special case of Set Cover:

Vertex Cover

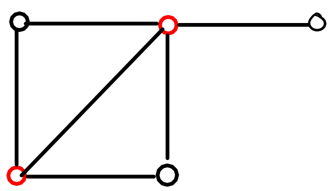
Input:

$G=(V,E)$

Objective:

Find a min. card. vertex set $C \subseteq V$ s.t. each edge $e \in E$ has at least one endpoint in C.

Ex:



With $\mathcal{S}=V$ and $\mathcal{E}=E$, Alg. 1 is a 2-approx. alg. for Vertex Cover.

One of the exercises for Tuesday: Write down LP for Vertex Cover.

Section 1.4: The dual LP

What is a dual?

Ex:

P

$$\begin{array}{ll} \min & 7x_1 + x_2 + 5x_3 \\ \text{s.t.} & x_1 - x_2 + 3x_3 \geq 10 \\ & 5x_1 + 2x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Primal

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq x_1 - x_2 + 3x_3 + 5x_1 + 2x_2 - x_3 \\ &\geq 10 + 6 = 16 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq 2(x_1 - x_2 + 3x_3) + 5x_1 + 2x_2 - x_3 \\ &\geq 2 \cdot 10 + 6 = 26 \end{aligned}$$

To find a largest possible lower bound on $7x_1 + x_2 + 5x_3$, we should determine y_1 and y_2 maximizing $10y_1 + 6y_2$, under the constraints that

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\overset{(*)}{\geq} y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{\geq 10} + y_2 \overbrace{(5x_1 + 2x_2 - x_3)}^{\geq 6} \\ &= \underbrace{(y_1 + 5y_2)}_{\leq 7} x_1 + \underbrace{(-y_1 + 2y_2)}_{\leq 1} x_2 + \underbrace{(3y_1 - y_2)}_{\leq 5} x_3 \end{aligned}$$

necessary to satisfy (*)

and $\underbrace{y_1, y_2, y_3 \geq 0}_{\text{otherwise}}$

„$\geq$" becomes „$\leq$"

Thus, we arrive at the following problem:

$$\boxed{D}$$

$$\boxed{\begin{array}{lrcl} \max & 10y_1 + 6y_2 & & \\ \text{s.t.} & y_1 + 5y_2 & \leq & 7 \\ & -y_1 + 2y_2 & \leq & 1 \\ & 3y_1 - y_2 & \leq & 5 \\ & y_1, y_2 \geq 0 & & \end{array}}$$

Dual

<u>In general:</u>

Primal:

$$\boxed{\begin{array}{ll} \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n \end{array}}$$

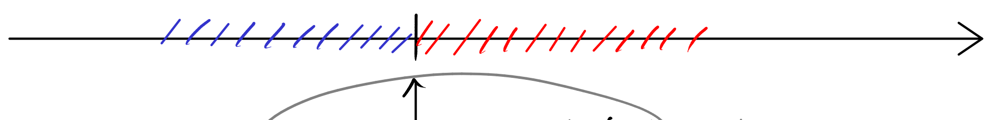
Dual:

$$\boxed{\begin{array}{ll} \max & b_1y_1 + b_2y_2 + \ldots + b_my_m \\ \text{s.t.} & a_{1j}y_1 + a_{2j}y_2 + \ldots + a_{mj}y_m \leq c_1, \quad j = 1, 2, \ldots, n \\ & y_i \geq 0, \quad i = 1, 2, \ldots, m \end{array}}$$

Returning to the example above:

The constraints of D ensure that the value of any sol. to D is a lower bound on the value of any sol. to P, i.e., for any pair x, y of sol. to P and D resp.,

$$10y_1 + 6y_2 \leq 7x_1 + x_2 + 5x_3$$ Weak duality



opt. value for both problems — Strong duality

Consider again the inequality leading to the constraints of the dual:

$y_1 = 0 \vee x_1 - x_2 + 3x_3 = 0 \Updownarrow$ $\qquad$ $\Updownarrow y_2 = 0 \vee 5x_1 + 2x_2 - x_3 = 6$

$$7x_1 + x_2 + 5x_3 \overset{(*)}{\geq} \overbrace{y_1 \underbrace{(x_1 - x_2 + 3x_3)}_{\geq 10}}^{= 10y_1} + \overbrace{y_2 \underbrace{(5x_1 + 2x_2 - x_3)}_{\geq 6}}^{= 6y_2}$$

$$= \underbrace{\overbrace{(y_1 + 5y_2)}^{\leq 7} x_1}_{= 7x_1} + \underbrace{\overbrace{(-y_1 + 2y_2)}^{\leq 1} x_2}_{= x_2} + \underbrace{\overbrace{(3y_1 - y_2)}^{\leq 5} x_3}_{= 5x_3}$$

$$\underbrace{\Updownarrow \; y_1 + 5y_2 = 7 \vee x_1 = 0 \qquad \Updownarrow \; -y_1 + 2y = 1 \vee x_2 = 0 \qquad \Updownarrow \; 3y_1 - y_2 = 5 \vee x_3 = 0}$$

$\Updownarrow$

(*) becomes =

Thus,

$$7x_1 + x_2 + 5x_3 = 10y_1 + 6y_2$$

$$\Updownarrow$$

Complementary Slackness Conditions

$$\left\{\begin{array}{lll} x_1 > 0 & \Rightarrow & y_1 + 5y_2 = 7 \\ x_2 > 0 & \Rightarrow & -y_1 + 2y_2 = 1 \\ x_3 > 0 & \Rightarrow & 3y_1 - y_2 = 5 \\ & & \\ y_1 > 0 & \Rightarrow & x_1 - x_2 + 3x_3 = 10 \\ y_2 > 0 & \Rightarrow & 5x_1 + 2x_2 - x_3 = 6 \end{array}\right.$$

The first three conditions: primal c.s.c.

The last two conditions: dual c.s.c.

By the Strong Duality Theorem (which we will not prove), there exist solutions fulfilling the c.s.c.

Moreover, if the c.s.c. are „close" to being satisfied, the values of the primal and dual sol. are „close":

Relaxed Complementary Slackness Conditions

$$\begin{cases} x_1 > 0 & \Rightarrow & y_1 + 5y_2 \geq 7/b \\ x_2 > 0 & \Rightarrow & -y_1 + 2y_2 \geq 1/b \\ x_3 > 0 & \Rightarrow & 3y_1 - y_2 \geq 5/b \\ & & \\ y_1 > 0 & \Rightarrow & x_1 - x_2 + 3x_3 \leq 10c \\ y_2 > 0 & \Rightarrow & 5x_1 + 2x_2 - x_3 \leq 6c \end{cases}$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq bc(10y_1 + 6y_2)$$

<u>Proof</u>:

$$(y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3$$

$$\geq \frac{7}{b}x_1 + \frac{1}{b}x_2 + \frac{5}{b}x_3, \text{ by the Primal relaxed c.s.c.}$$

$$= \frac{1}{b}(7x_1 + x_2 + 5x_3)$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq b\big((y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3\big)$$

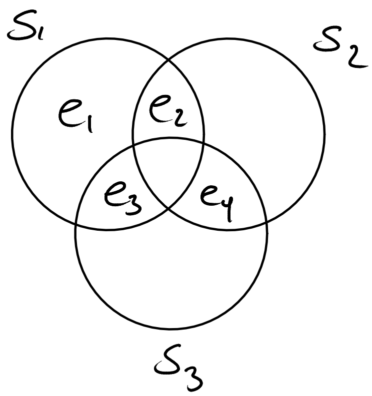
$$= b\big((x_1 - x_2 + 3x_3)y_1 + (5x_1 + 2x_2 - x_3)y_2\big)$$

$$\leq b(10cy_1 + 6cy_2), \text{ by the Dual r.c.s.c.}$$

$$= bc(10y_1 + 6y_2)$$

What is the dual of the Set Cover LP?

Ex:



$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

Primal:

$$
\begin{array}{llcl}
\min & x_1 + 2x_2 + 3x_3 & & \\
\text{s.t.} & x_1 & \geq & 1 \\
 & x_1 + x_2 & \geq & 1 \\
 & x_1 + x_3 & \geq & 1 \\
 & x_2 + x_3 & \geq & 1 \\
 & x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

OPT = 3:

$x_1 = x_2 = 1$

Dual:

$$
\begin{array}{llcl}
\max & y_1 + y_2 + y_3 + y_4 & & \\
\text{s.t.} & y_1 + y_2 + y_3 & \leq & 1 \\
 & y_2 + y_4 & \leq & 2 \\
 & y_3 + y_4 & \leq & 3 \\
 & y_1, y_2, y_3, y_4 \geq 0 & &
\end{array}
$$

OPT = 3:

$y_1 = 1$, $y_4 = 2$ or $y_3 = 1$, $y_4 = 2$

Set Cover Primal

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \sum_{j: e_i \in S_j} x_j \geq 1, \quad i = 1, 2, \ldots, n$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, m$$

Covering problem

Set Cover Dual

$$\max \sum_{i=1}^{n} y_i$$

$$\text{s.t.} \sum_{e_i \in S_j} y_i \leq w_j, \quad j = 1, 2, \ldots, m$$

$$y_i \geq 0, \quad i = 1, 2, \ldots, n$$

Packing problem

Recall that the dual is constructed such that the value of any solution to the dual is a lower bound on the value of any solution to the primal:

$$Z_{Primal} \geq Z_{Dual} \quad \text{(weak duality property)}$$

In fact,

$$Z^*_{Primal} = Z^*_{Dual} \quad \text{(strong duality property)}$$