## Set Cover - recap.

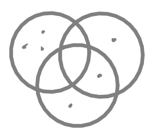
LP-relax: $\min \sum_{j=1}^{m} x_j w_j$

$$\text{s.t.} \sum_{j:\, e_i \in S_j} x_j \geq 1, \qquad 1 \leq i \leq n$$

$$x_j \geq 0, \qquad 1 \leq j \leq m$$

Deterministic rounding: $f$-approx. alg.

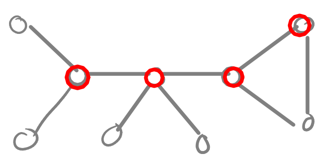
Eks:



$f = 2$

Vertex Cover

Ex:



## Exercises

## Recap ctd.:

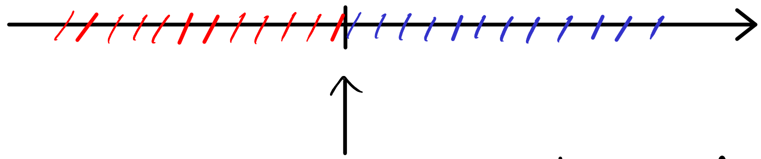
Dual LP: $\max \sum_{i=1}^{n} y_i$

$$\text{s.t.} \sum_{e_i \in S_j} y_i \leq w_j, \qquad 1 \leq j \leq m$$

$$y_i \geq 0, \qquad 1 \leq i \leq n$$

For <u>any</u> pair $\vec{x}, \vec{y}$ to the primal / dual problems:

$$\sum_{i=1}^{n} y_i \leq \sum_{j=1}^{m} x_j w_j \qquad \text{(weak duality)}$$



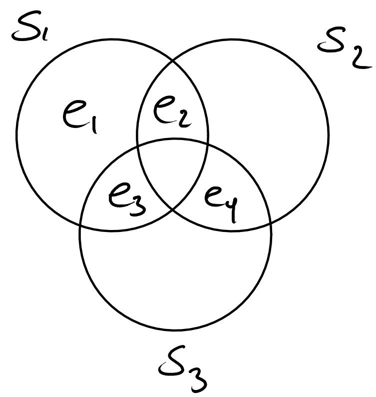
optimum objective value for both problems (strong duality)

(Relaxed) Complementary Slackness Conditions:

Primal c.s.c. $\quad x_j > 0 \Rightarrow \sum_{e_i \in S_j} y_i = w_j \quad \left(\geq \frac{1}{b} w_j\right), \quad 1 \leq j \leq m$

Dual c.s.c. $\quad y_i > 0 \Rightarrow \sum_{j : e_i \in S_j} x_j = 1 \quad (\leq c), \quad 1 \leq i \leq n$

Ex:




$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

## Primal:

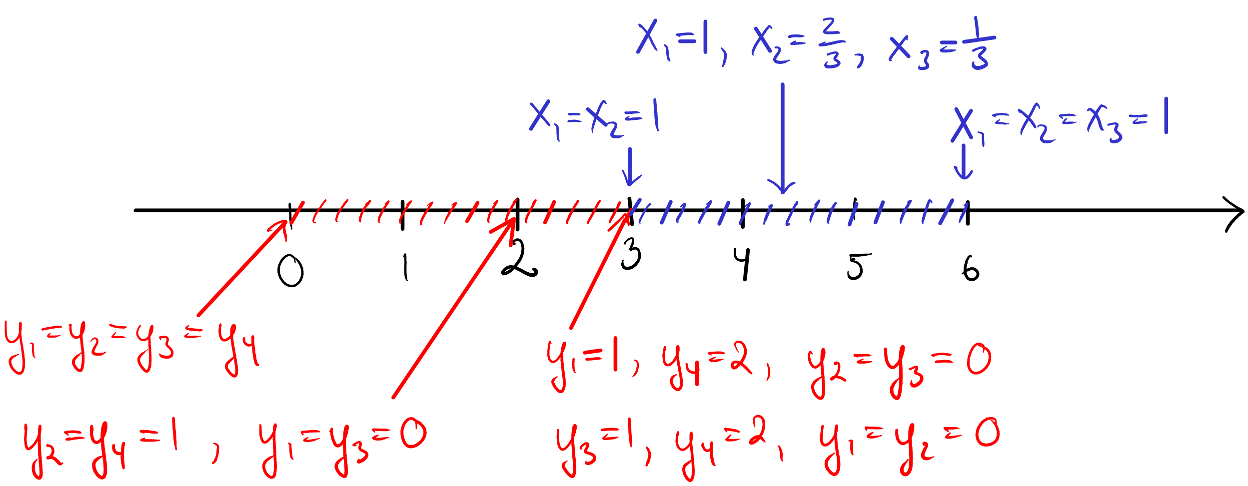
$$\begin{array}{ll} \min & x_1 + 2x_2 + 3x_3 \\ \text{s.t.} & x_1 \geq 1 \\ & x_1 + x_2 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

OPT = 3:
$x_1 = x_2 = 1$

## Dual:

$$\begin{array}{ll} \max & y_1 + y_2 + y_3 + y_4 \\ \text{s.t.} & y_1 + y_2 + y_3 \leq 1 \\ & y_2 + y_4 \leq 2 \\ & y_3 + y_4 \leq 3 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

OPT = 3:
$y_1 = 1$, $y_4 = 2$ or $y_3 = 1$, $y_4 = 2$



## Alg. 2 for Set Cover

$\vec{y}^* \leftarrow$ opt. sol. to dual LP
$I' \leftarrow \{ j \mid \sum_{e_i \in S_j} y_i = w_j \}$

In the ex. above:

with $y_1^* = 1$, $y_4^* = 2$, Alg 2 would choose $S_1$ and $S_2$ with a total weight of 3.

with $y_3^* = 1$, $y_4^* = 2$, Alg. 2 would choose $S_1, S_2$, and $S_3$ with a total weight of 6.

The first solution is optimal, and the latter is a 2-approximation (i.e., an $f$-approximation).

Alg. 2 is an $f$-approximation algo.:

If the algo. chooses $S_1, S_2$, and $S_3$, the total weight is $W = w_1 + w_2 + w_3$, and

$$w_1 + w_2 + w_3 = (y_1^* + y_2^* + y_3^*) + (y_2^* + y_4^*) + (y_3^* + y_4^*),$$

since the algo. chooses exactly those sets that have LHS = RHS.

Since each $y_i$ is present in at most $f$ constraints,

$$\begin{aligned} W &\leq f \cdot (y_1^* + y_2^* + y_3^* + y_4^*) \\ &= f \cdot Z^*_{dual} \\ &\leq f \cdot Z^*_{primal}, \text{ by the weak duality property} \\ &= f \cdot OPT \end{aligned}$$

Lemma 1.7

Alg. 2 produces a set cover

Proof:

Assume for the sake of contradiction that some element $e_k$ is not covered by $\{S_j \mid j \in I'\}$.

Then $\sum_{e_i \in S_j} y_i < w_j$ for all $S_j$ containing $e_k$.

These are exactly the constraints involving $y_k$.

Thus, none of the constraints involving $y_k$ are tight.

This means that $y_k$ can be increased without violating any constraint.

Since this will increase the value $\sum_{i=1}^{n} y_i$ of the sol., we conclude that

the solution $\vec{y}$ was not optimal. $\square$

Ex:

In the ex. above, assume

$y_1 = y_2 = y_3 = 0$

$y_4 = 2$

Then, only the second constraint is tight, so only $S_2$ is picked:

$$
\begin{aligned}
y_1 + y_2 + y_3 \qquad &= 0 < 1 \\
y_2 \qquad + y_4 &= 2 \\
y_3 + y_4 &= 2 < 3
\end{aligned}
$$

$e_4$ is not covered, since none of the two constraints involving $y_4$ are tight.

We can increase $y_3$ from 0 to 1 without violating any constraints
(Then two other constrainsts become tight.)
This increases the sol. value from 2 to 3.
Thus, the sol. above was not optimal.

Or we could increase $y_1$ from 0 to 1.
Then only the first constraint becomes tight, resulting in an optimal solution.

This illustrates the idea of the primal-dual alg of Section 1.5.

We now give a more formal proof that Alg 2 is an $f$-approximation algo.

**Thm 1.8**

Alg. 2 is an $f$-approx. algo.

<u>Proof</u>:

The correctness follows from Lemma 1.7.

Approx. guarantee:

$$\sum_{j \in I'} w_j = \underbrace{\sum_{j \in I'} \sum_{e_i \in S_j}}_{y_i^* \text{ appears once for each set in the sol. containing } e_i} y_i^*$$

$$= \sum_{i=1}^{n} \underbrace{\left| \{ j \in I' \mid e_i \in S_j \} \right|}_{\#\text{sets in the sol. containing } e_i} \cdot y_i^*$$

$$\leq \sum_{i=1}^{n} \underbrace{f_{e_i}}_{\#\text{sets containing } e_i} \cdot y_i^*$$

$$\leq \sum_{i=1}^{n} f \cdot y_i^*$$

$$= f \cdot Z_{dual}^*$$

$$\leq f \cdot Z_{primal}^*, \text{ by the weak duality property}$$

$$\leq f \cdot OPT$$

$\square$

Note that for proving the above theorem, we could also use the relaxed c.s.c. (with $b=1$, $c=f$), since

$$\sum_{j: e_i \in S_j} x_j \le f, \quad \text{for all } i=1,2,\dots,n$$

Note that, on any instance of Set Cover, $I \subseteq I'$:

Since the LP is solved optimally,

$x_j > 0 \Rightarrow$ constraint $j$ is tight $\Rightarrow j \in I'$.

Thus, $j \in I \Rightarrow x_j \geq \frac{1}{f} \Rightarrow j \in I'$

Thus, Alg. 1 is always at least as good as Alg. 2.

Both Alg. 1 and Alg. 2 rely on solving an LP (optimally). In Section 1.5, we will study a more time efficient alg.

The key observation is that in the proof of Thm 1.8, we did not need the fact that $\vec{y}^*$ is optimal, since $Z_{dual} \leq Z^*_{primal}$, for any feasible dual solution.

Thus, the crux is to obtain an index set $I''$ s.t.

- $\bigcup_{j \in I''} S_j$ is a vertex cover
- $\sum_{j \in I''} w_j = \sum_{j \in I''} \sum_{e_i \in S_j} y_i$, for some feasible sol. $\vec{y}$ to the dual LP

without solving an LP optimally.

# Section 1.5 : A Primal-Dual Alg. for Set Cover

**Alg. 1.1 for Set Cover : Primal-Dual**

$I'' \leftarrow \emptyset$

$\vec{y} \leftarrow \vec{0}$

While $\exists e_k \notin \bigcup_{j \in I''} S_j$

  Increase $y_k$ until some constraint, $\ell$, becomes tight, i.e., $\sum_{e_i \in S_\ell} y_i = w_\ell$

  $I'' \leftarrow I'' \cup \{\ell\}$

Note that $e_k \in S_\ell$

**Thm 1.9**

Alg. 1.1 is an $f$-approx. alg. for Set Cover

Proof:

Alg. 3 produces a set cover, since as long as some element is not covered, the corresponding dual constraints are non-tight.

The approx. guarantee follows from the same calculations as in the proof of Thm. 1.8, since

$$\sum_{j \in I''} w_j = \sum_{j \in I''} \sum_{e_i \in S_j} y_i \leq f \cdot Z_{dual} \leq f \cdot Z^*_{dual}$$

$\square$

In contrast to Alg. 2 from Section 1.4, Alg. 1.1 does not necessarily produce an optimal dual solution:
In the example above, it might do the following.

$y_2 \leftarrow 1$ ($S_1$ is picked, $e_4$ still uncovered)
$y_4 \leftarrow 1$ ($S_2$ is picked)

(This is fine, since the proof of Thm. 1.8 does not use that $\sum y_i = OPT$, only that $\sum y_i \leq OPT$, which is true for <u>any</u> feasible sol. to the dual.)