Primal-dual recap.

Primal-dual alg

Create a feasible dual sol., e.g., $\vec{y} \leftarrow \vec{0}$
(Based on the dual solution,) create an (infeasible)
primal solution, e.g., $\vec{x} \leftarrow \vec{0}$
While the primal solution is infeasible
Modify dual solution to increase dual obj. value, maintaining feasibility
Modify primal solution „accordingly"

Primal-dual for Set Cover

For Set Cover, we increased a dual variable corresponding to an uncovered element, until a constraint became tight.
Then, we picked the corresponding set.

Analysis

(a) Correctness:

As long as some element e is uncovered, all constraints containing y_e are nontight.

(b) Approx.:

The resulting primal obj. value is a sum of optimal dual variables, where each y_i^* appears $\leq f$ times

Alternative analysis of (b)

Recall that (b) follows from the fact that y_e appears only in constraints corresponding to sets containing e and that there are at most f such sets.

Note that, similarly, each constraint in the primal has at most f terms.

This trivially implies that we fulfill the relaxed dual c.s.c. with $c = f$.

Since we only choose sets corresponding to tight dual constraints, we also fulfill the primal c.s.c. $(b=1)$.

Thus, we could also use the relaxed c.s.c. to prove that the alg. is an f-approx. alg.

Section 1.7: Randomized Rounding

Alg RR_1

```
Solve LP
I ← ∅
For j ← 1 to m
    With probability x_j
        I ← I ∪ {j}
```

Expected cost $= Z^*_{LP} \leq OPT$, but the result is most likely <u>not</u> a set cover.

Alg RR_2

```
Solve LP
  I ← ∅
  For i ← 1 to 2·ln(n)
    For j ← 1 to m
        With probability x_j
            I ← I ∪ {j}
```

Expected cost $\leq 2 \cdot \ln(n) \cdot Z^*_{LP} \leq 2 \cdot \ln(n) \cdot OPT$, and high probability that all elements are covered.

(Calculations below)

Alg RR_3

Solve LP

Repeat

$I \leftarrow \emptyset$

For $i \leftarrow 1$ to $2 \cdot \ln(n)$

For $j \leftarrow 1$ to m

With probability x_j

$I \leftarrow I \cup \{j\}$

Until $\{S_j \mid j \in I\}$ is a set cover and $w(I) \leq 4 \ln(n) Z^*_{LP}$

Cost $\leq 4 \cdot \ln(n) \cdot OPT$

Result is a set cover.

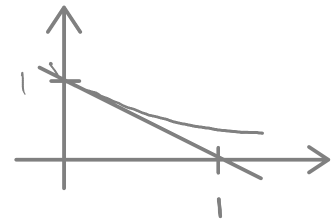
Expected running time is polynomial.

(Calculations below)

p_i: prob. that e_i is covered

$\overline{p_i} = 1 - p_{k(i)}$: prob. that e_i is <u>not</u> covered

Alg RR_1:



$$\bar{p}_i = \prod_{j:\, e_i \in S_j} \overbrace{(1-x_j)}^{\le e^{-x_j},\ \text{for any } x_j \in \mathbb{R}}$$

$$\le \prod_{j:\, e_i \in S_j} e^{-x_j}$$

$$= e^{\underbrace{-\sum_{j:\, e_i \in S_j} x_j}_{\le -1,\ \text{by the LP constraint corresponding to } e_i}}$$

$$\le e^{-1}$$

Alg RR_2:

$$\bar{p}_i = (\bar{p}_i)^{2\ln n} \le e^{-2\ln n} = \left(e^{-\ln n}\right)^2 = n^{-2}$$

$$Pr[\text{not set cover}] \le \sum_{i=1}^{n} \bar{p}_i \le \sum_{i=1}^{n} n^{-2} = n \cdot n^{-2} = n^{-1}$$

$$\underbrace{Pr[w(I) \geq 4 \cdot \ln(n) \cdot z^*_{LP}]}_{>\frac{1}{2} \text{ would give } E[w(I)] > 2 \cdot \ln(n) \cdot z^*_{LP}\ ↯} \le \frac{1}{2}, \text{ by Markov's Inequality:}$$

Alg RR_3:

$$Pr[\text{„not set cover" or „too expensive"}] \le n^{-1} + \frac{1}{2}$$

Thus,

$$E[\#\,\text{iterations}] \le \frac{1}{1-(n^{-1}+\frac{1}{2})} \approx 2$$

Sometimes randomized algorithms are simpler / easier to describe / come up with.
Sometimes randomized algorithms can be derandomized as we saw in Chapter 5.
Exercise for Tuesday: derandomize $\mathrm{Alg}\,RR_3$ (Ex. 5.7)

Section 1.6: A Greedy Algorithm

A natural greedy choice would be to „pay" as little as possible for each additional covered element:

Alg 1.2 for Set Cover: Greedy

$I \leftarrow \emptyset$

For $j \leftarrow 1$ to m

  $\hat{S}_j \leftarrow S_j$ (uncovered part of S_j)

While $\{S_j \mid j \in I\}$ is not a set cover

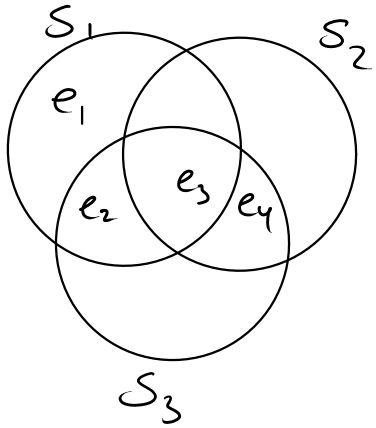
  $\ell \leftarrow \arg\min\limits_{j : \hat{S}_j \neq \emptyset} \dfrac{w_j}{|\hat{S}_j|}$ (S_ℓ: set with smallest cost per uncovered element)

  $I \leftarrow I \cup \{\ell\}$

  For $j \leftarrow 1$ to m

    $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$

Ex:



$w_1 = 12$

$w_2 = 4$

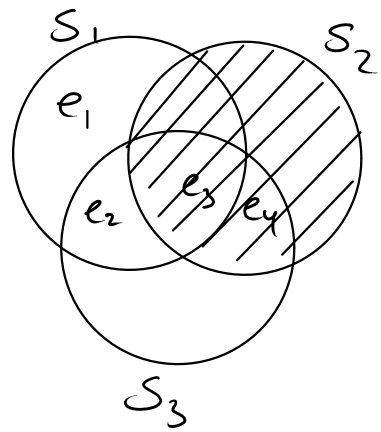
$w_3 = 9$

$\frac{w_1}{|S_1|} = \frac{12}{3} = 4,$

$\frac{w_2}{|S_2|} = \frac{4}{2} = 2$ ← price per element in first iteration

$\frac{w_3}{|S_3|} = \frac{9}{3} = 3$

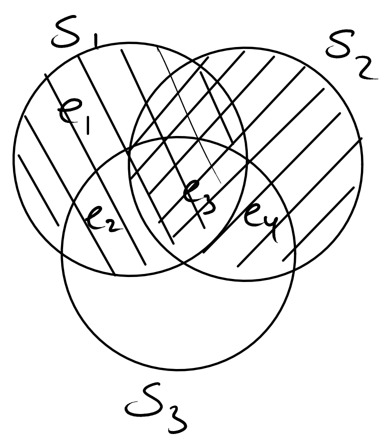
→ Pick S_2



$\frac{w_1}{|\hat{S}_1|} = \frac{12}{2} = 6$ ← price per element in second iteration

$\frac{w_3}{|\hat{S}_3|} = \frac{9}{1} = 9$

→ Pick S_1



$$\text{Total weight} = \sum_{i=1}^{4} \text{price}(e_i) = 2+2+6+6$$

$$= w_2 + w_1 = 4 + 12$$

$$= 16$$

The greedy alg. is an H_n-approx. alg

Recall: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \approx \ln(n)$

It is „likely" that no significantly better approx. ratio can be obtained:

Thm 1.13:

Approx. factor $\frac{\ln n}{c}$, $c > 1$, for unweighted Set Cover

$\Rightarrow \underbrace{n^{O(\log \log n)}}_{\sim k^{\log n}}$ - approx alg. for NPC

Thm 1.11

Alg. 1.2 is an H_n-approx. alg. for Set Cover

Proof:

n_k: # uncovered elements at the beginning of the k'th iteration

In the ex. above:

$n = 4$

$n_1 = 4, \quad n_2 = 2, \quad n_3 = 0$

$n_1 - n_2 = 2, \quad n_2 - n_3 = 2$

Any algorithm, including OPT, has to cover these n_k elements using only sets in $\mathcal{S} - \{S_j \mid j \in I\}$, since none of them are contained in $\{S_j \mid j \in I\}$.

Hence, there must be at least one element with a price of at most OPT/n_k. Otherwise, OPT would not be able to cover the n_k elements (and certainly not all n elements) at a cost of only OPT.

Hence, the $n_k - n_{k+1}$ elements covered in iteration k cost at most $(n_k - n_{k+1})\, OPT/n_k$ in total.

Thus, the cost of the set cover produced by the greedy alg. is

$$\sum_{j \in I} w_j \leq \sum_{k=1}^{r} \frac{n_k - n_{k+1}}{n_k} OPT$$

\# iterations

$$= OPT \sum_{k=1}^{r} (n_k - n_{k+1}) \cdot \frac{1}{n_k}$$

$$\leq OPT \sum_{k=1}^{r} \Big(\underbrace{\frac{1}{n_k} + \frac{1}{n_k - 1} + \dots + \frac{1}{n_{k+1}+1}}_{n_k - n_{k+1} \text{ terms that are each } \geq \frac{1}{n_k}} \Big)$$

$$= OPT \sum_{s=1}^{n} \frac{1}{s}$$

$$= OPT \cdot H_n \qquad \square$$

Ex from before:

$OPT = w_1 + w_2 = 12 + 4 = 16$

The cost of the greedy alg is

$$w_2 + w_1 = 4 + 12$$

$$= 2 + 2 + 6 + 6$$

$$\leq \left(\frac{16}{4} + \frac{16}{4}\right) + \left(\frac{16}{2} + \frac{16}{2}\right)$$

$$\leq \left(\frac{16}{4} + \frac{16}{3}\right) + \left(\frac{16}{2} + \frac{16}{1}\right)$$

$$= 16 \cdot \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right)$$

$$= 16 \cdot H_4$$